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Article

New Class of Ideal Topological Spaces

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Abstract: In this study, we establish a newly constructed operators W_Δ for establishing a new kind of sets entitled $W_\Delta - K$ sets. By preserving both local and transitional qualities in a topological space, these sets generalize and improve a number of traditional and generalized topological constructions. We examine the structural properties of $W_\Delta - K$ sets in comparison to pre-open forms as well as semi-regular closed sets. We show that the $W_\Delta - K$ set represents strictly weaker compared to the α-open sets in general. Additionally, we demonstrate that all $W_\Delta - K$ sets constitute a supratopology by showing that their collection occurs under arbitrary unions. The newly developed family of sets offers fresh insights into continuity, closure, and convergent analysis as well as a fundamental basis for creating sophisticated ideas in extended topological spaces. These results offer a theoretical structure that may be expanded to examine intricate connections between different generalized open sets, opening up new avenues for sophisticated topological modeling applications.Furthermore, we demonstrate that the collection of all -K sets forms a supratopology, as it is closed under arbitrary unions. Theintroduction This is a new family of sets.provides fresh perspectives on continuity, closure operations, and convergence analysis, offering a robust framework for developing advanced notions in extended topological structures. The findings presented in this work open new avenues for exploring intricate relationships among various generalized open sets and pave the way for sophisticated modeling applications in modern topology.

Keywords: $W_\Delta - K$ set, Ideal topological spaces, Local function continuity closure operator

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1. Introduction

Following the foundational introduction of the concepts of ideal and local function by Kuratowski [1], considerable attention has been devoted to these topics within the topological literature. A significant development in this domain occurred in 1986 when Natkaniec introduced the Ψ-set operator [2], which subsequently gave rise to a range of generalizations including Ψ-sets [3], Ψ–C sets [4], $*\Psi$-sets [5], and $\Psi*$-sets [6], all formulated within the framework of the Ψ-operator. Building on this line of inquiry, Al-Omari and Noiri explored the local closure function in conjunction with the Ψ_Γ-operator in the context of ideal topological spaces, through which they generated novel topological structures [7]. Further contributions were made by Islam and Modak, who introduced the semi-closure local function and constructed a new topology based on it [8].

Additionally, Pavlović investigated the specific conditions under which the local function and the local closure function coincide [9]. This stream of research was extended by Tunç and Özen Yıldırım, who introduced and analyzed new classes of sets such as I_Δ-dense sets, Δ -dense-in-itself sets, and I_Δ-perfect sets by leveraging the local closure function [10].

In the present study, we propose the concept of the $W_\Delta - K$ set, generated through the W_Δ -operator. We rigorously examine the interrelations of $\Omega_\Delta - R$ sets with various existing generalized sets including LΓ-perfect, RΓ-perfect, IΓ-perfect sets, and others studied in previous literature [11]. Furthermore, we explore key structural properties of $W_\Delta - K$ sets and derive several new results that enrich the current understanding of ideal-based topological constructs.

2. Materials and Methods

In this study, we construct the $W\Delta$WΔ-operator and define the corresponding class of $W\Delta$WΔ-K sets. First, we establish the basic definition of the operator within an ideal topological setting. Then, we compare the newly defined sets with several generalized structures such as $L\Gamma$LΓ-perfect, R Γ RΓ-perfect, I ΓIΓ-perfect, and Ω Δ Ω Δ-R sets to determine their relationships.We analyze the main structural properties of $W\Delta$WΔ-K sets, including their behavior under unions and intersections, and demonstrate that they form a supratopology. Based on these analyses, we derive new results that extend current theories related to ideal-based and operator-induced topological sets.

3. Results and Discussion

1. Preliminaries

1. Let (X, τ) denote a topological space, or X for short. In this context, the closure and interior of any subset K ⊆ X are denoted by cl(K) and int(K), respectively [12]. The power set of X, represented by P(X), is the set of all subsets of X.
2. An ideal I on X is a non-empty set of subsets of X that satisfies the following two conditions:Heredity: If $\mathcal{K} \in I$ *and* $\mathcal{P} \subseteq \mathcal{K}$, then $\mathcal{P} \in I$,
3. Finite additivity: If $\mathcal{K}, \mathcal{P} \in I$, then $\mathcal{K} \cup \mathcal{P} \in I$.

   A topological space with an ideal, or an ideal topological space, is denoted by (X, τ, I) where I is an ideal on the topological space (X, τ).

   For any subset $\mathcal{K} \subseteq X$, the local function of $\mathcal{K}$ with respect to τ and I, denoted by $\mathcal{K}^*$, is defined as:

$\mathcal{K}^* = \{x \in X \mid \cap \mathcal{K} \in \mathrm{I}\ \forall\ G \in \tau(x)\}$,	(1) (1) ())())

where $\tau(x) = \{G \in \tau \mid x \in G\}$ is the neighborhood filter at x.
Similarly, the local closure function [13] of $\mathcal{K}$ with respect to I and τ, denoted by $\Gamma(\mathcal{K})$, is defined as:

$\Gamma(\mathcal{K}) = \{x \in X \mid \mathcal{K} \cap cl(G) \in \mathrm{I}\ \ \forall\ G \in \tau(x)\}$.	(2) (1) ())())

Using the local function, Natkaniec introduced the operator Ψ defined for each $\mathcal{K} \in P(X)$ as: $\Psi(\mathcal{K}) = X \setminus (X \setminus \mathcal{K})^*$. Further, Al-Omari and Noiri introduced another operator $\Psi_\Gamma: P(X) \to \tau$, defined as: $\Psi_\Gamma(\mathcal{K}) = X \setminus \Gamma(X \setminus \mathcal{K})$, for each $\mathcal{K} \subseteq X$. Based on these operators, we introduce a various types of generalized sets have been studied[14]:

- A subset $\mathcal{K} \subseteq X$ is called I_Δ-perfect if $\Gamma(\mathcal{K}) = \mathcal{K}^c$,
- Δ -dense-in-itself if $\mathcal{K} \subseteq \Gamma(\mathcal{K}^c)$,
- L_Δ -perfect if $\mathcal{K} \setminus \Gamma(\mathcal{K}^c) \in I$,
- R_Δ -perfect if $\Gamma(\mathcal{K}) \setminus \mathcal{K}^c \in I$,
- I_Δ -dense if $\Gamma(\mathcal{K}^c) = X$.

Moreover, a subset $\mathcal{K}$ of X is said to be a Ψ–C set [5] if it satisfies the condition:

$\mathcal{K} \subseteq cl(\Psi(\mathcal{K}))$.	(3) (1) ())())

Let (X, τ) be a topological space and $\mathcal{K} \subseteq X$ a subset. The θ-closure of $\mathcal{K}$, denoted by cl_θ $(\mathcal{K})$, is defined as: $cl_\theta(\mathcal{K}) = \{x \in X \mid cl(G) \cap \mathcal{K} \neq \emptyset\ \forall\ G \in \tau(x)\}$, as introduced in [15]. This concept captures the idea of closure with respect to neighborhoods whose closures intersect the set $\mathcal{K}$.
Correspondingly, the θ-interior of $\mathcal{K}$, denoted by $int_\theta(\mathcal{K})$, consists of all points $x \in \mathcal{K}$ for which there exists an open set $G \subseteq \tau$ such that $x \in G, G \subseteq cl(G)$, and $cl(G) \subseteq \mathcal{K}$ [16].
A subset $\mathcal{K}$ of X is called θ-closed if $\mathcal{K} = cl_\theta(\mathcal{K})$. The complement of a θ-closed set is referred to as a θ-open set, and the collection of all θ-open sets in (X, τ) is denoted by τ_θ , which itself forms a topology on X.
In , Al-Omari and Noiri introduced two further topologies on X based on the operator Ψ_Γ :

$\omega = \{\mathcal{K} \subseteq X \mid \mathcal{K} \subseteq \Psi_\Gamma(\mathcal{K})\}$	(4) (1) ())())
$\omega_0 = \{\mathcal{K} \subseteq X \mid \mathcal{K} \subseteq int(cl(\Psi_\Gamma(\mathcal{K})))\}$.	(5) (1) ())())

These satisfy the inclusion relation: $\omega_0 \subseteq \omega \subseteq \omega_0$.
A set $\mathcal{K}$ considered to be σ-open (respectively, ω_0-open) if $\mathcal{K} \in \omega$ (respectively, $\mathcal{K} \in \omega_0$).
Further generalizations include:

- A subset $\mathcal{K}$ is θ-I-closed [16] if $\Gamma(\mathcal{K}) \subseteq \mathcal{K}$,
- $\mathcal{K}$ is called regular θ-closed [17] if $\mathcal{K} = cl_\theta(int_\theta(\mathcal{K}))$,
- $\mathcal{K}$ is said to be semi θ-open if $\mathcal{K} \subseteq cl_\theta(int_\theta(\mathcal{K}))$,
- $\mathcal{K}$ is θ-semiopen if there exists a θ-open set G such that $G \subseteq \mathcal{K} \subseteq cl(G)$. The family of all such θ-semiopen sets is denoted by $SO_\theta(X, \tau)$.

Moreover:
A subset K is termed M*-open if K⊆int(cl(int_θ(K))).
A set K is preopen if K⊆int(cl(K)), and its complement is called preclosed.
Finally, K is generalised closed (or g-closed) if Gcl(K)⊆G whenever K⊆G and G is open.
Let Y be a non-empty set and let τ_0 be a collection of subsets of Y.
If Y ∈ τ_0 and τ_0 is closed under arbitrary unions, then τ_0 is called a supratopology on Y and the pair (Y, τ_0) is referred to as a supratopological space or simply a supraspace.
Some properties of W_Δ-K sets
Definition 3.1. Let (X, τ, I) be an ITS and let K ⊆ X. Then, for each K ⊆ X, define a mapping W_Δ: P(X) → τ by: W_Δ(K) = X \ {Γ(K^c)}^c, where K ⊆ X, is considered to be the W-K operator. Furthermore, K⊆X represents a W-K set if it satisfies the condition K⊆cl(Ψ(K))^c.
Definition 3.2. Assume that (X, τ, I) is an ITS and that K ⊆ X. A set K is considered to be a W_Δ-K set if K ⊆ cl(Ω_Δ(K)). The compilation of all the W_Δ-K sets in (X, τ, I) is indicated by Ω_Δ(X, τ, I).
Theorems 3.3. Let (X, τ, I) be an ITS and let K ⊆ X. Then int_θ(K) ⊆ W_Δ(K).
Proof. Let K be an arbitrary subset of X in the ITS (X, τ, I). Suppose, by way of contradiction, that there exists a point x in X such that x ∉ W_Δ(K). By the definition of the operator W_Δ, this implies that x belongs to the complement of the local closure function, i.e. x belongs to ∅(K^c). By definition of the local closure function, it follows that, for every neighbourhood G in the neighbourhood system τ(x), the intersection cl(G)∩(K^c) is not an element of the ideal I; that is, cl(G)∩(K^c)∉I. Consequently, cl(G)∩K = ∅. for all $G \in \tau(x)$, meaning no open set around x, has a closure entirely contained in $\mathcal{K}$. Therefore, $x \notin int_\theta(\mathcal{K})$. Thus, every point not in $W_\Delta(\mathcal{K})$ is also not in $int_\theta(\mathcal{K})$, implying $int_\theta(\mathcal{K}) \subseteq W_\Delta(\mathcal{K})$, as required.
Theorems 3.4: Let (X, τ, I) be an ITS. If K belongs to τ_θ, then K is a W_Δ-K set.

Proof: Assume that K belongs to τ_θ, i.e. that K is a θ-open set. We know that K⊆W_Δ(K).

Since W_Δ(K)⊆cl(W_Δ(K)), it follows that K⊆cl(W_Δ(K)). Therefore, by definition, K is a W_Δ-K set [18].

Remark 3.5: In an ITS, not every open set is necessarily a W_Δ-K set. In other words, being open in the classical topological sense does not guarantee inclusion in the closure of its image under the W_Δ operator.

Example 3.6: Consider the finite set X = {h, g, f, e} with the topology τ = {∅, {h}, {e}, {h, g}, {h, e}, {h, g, e}, X} and the ideal I = {∅, {h}}. (6) (1) ()))))

Let A = {h}, which is clearly an open set in τ; however, a direct computation shows that cl(W_Δ(K)) = ∅. Therefore, K is not contained in the closure of W_Δ(K) and thus fails to be a W_Δ-K set despite being open. This illustrates that openness alone does not ensure the W_Δ-K property in ideal topological spaces.

(See Theorem 3.7.) Let (X, τ, I) be an ITS. If a subset K belongs to τ_θ, then K is a W_Δ-K set [19].

Proof. Suppose K belongs to τ_θ, i.e. K is θ-open. It follows that K⊆W_Δ(K). Since W_Δ(K) is always contained within its closure, we have K⊆cl(W_Δ(K)). According to the definition of a W_Δ-K set, this inclusion implies that K is a W_Δ-K set.

Remark 3.8: In the context of ITS, a W_Δ-K set is not necessarily θ-open, nor even open in the topological sense.Example

3.9. Let X = {h, g, f, e} be a finite set with the topology T = {∅, {e}, {h, f}, {h, e}, {h, f, e}, X} and the ideal I = {∅, {g}, {g, f}}. Take K = {e}. Then W_Δ(K) = {g} and cl(W_Δ(K)) = {g, e}. Therefore, K⊆cl(W_Δ(K)), meaning that K is a W_Δ-K set. However, K is neither open nor θ-open in this space.

Proposition 3.10: Let (X, τ, I) be an ITS and let K ⊆ X. If K^c is θ-closed, then K is a W_Δ-K set.

Proof. Assume that K^c is θ-closed in the ITS (X, τ, I). By the definition of θ_I-closed sets, we have that Γ(K^c) ⊆ K^c. Taking the complement of both sides yields K⊆X\Γ(K^c) = W_Δ(K). Since W_Δ(K)⊆cl(W_Δ(K)), it follows that K⊆cl(W_Δ(K)), meaning that K satisfies the definition of a W_Δ-K set [20].

Remark 3.11: The converse of Proposition 3.10 does not generally hold; that is to say, a W_Δ-K set does not necessarily imply that its complement is θ-closed.

Example 3.12: Let X = {h, g, f, e} be a finite set with topology T = {∅, {h}, {h, f}, {h, f, e}, X} and ideal I = {∅}. Consider the subset K = {h, g}. It can be shown by straightforward calculation that K⊆cl(W_Δ(K)) and therefore that K is a W_Δ-K set. However, its complement, K^c = {f, e}, is not θ_I-closed. This example confirms that the converse of the theorem does not generally hold [21].

Theorem 3.13: Let (X, τ, I) be an ITS and let K ⊆ X.

If K is a W_Δ-K set and W_Δ(K) is closed, then K^c is θ_I-closed.

Proof. Assume that K is a W_Δ-K set in (X, τ, I), meaning that K ⊆ cl(W_Δ(K)).

Since W_Δ(K) is closed, cl(W_Δ(K)) = W_Δ(K), so K ⊆ W_Δ(K).

Recall that W_Δ(K)=X\Γ(K^c). Therefore, K⊆X\Γ(K^c) implies that Γ(K^c)⊆K^c, which directly implies that K^c is θ_I-closed.

(See Theorem 3.14.) Let (X, τ, I) be an ITS and let K ⊆ X. If K^c is I_Δ-perfect, then K is a W_Δ-K set.

Proof.

Suppose that $\mathcal{K}^c$ is an I_Δ-perfect subset of X, meaning that $\Gamma(\mathcal{K}^c) = \mathcal{K}^c$. Taking complements on both sides, we obtain $\mathcal{K} = X \setminus \Gamma(\mathcal{K}^c) = W_\Delta(\mathcal{K})$. Since $W_\Delta(\mathcal{K}) \subseteq cl(W_\Delta(\mathcal{K}))$, it follows that $\mathcal{K} \subseteq cl(W_\Delta(\mathcal{K}))$, which implies that $\mathcal{K}$ is a $W_\Delta - K$ set.

Remark 3.15. The converse of Theorem 3.14 does not hold in general; that is, a subset $\mathcal{K}$ being a $W_\Delta - K$ set does not necessarily imply that its complement $\mathcal{K}^c$ is I_Δ-perfect.

Example 3.16: Let X = {h, g, f, e}, τ = {∅, {h}, {e}, {h, f}, {h, e}, {h, f, e}, X} and I = {∅, {h}}. In ITS(X, τ, I), consider the subset A = {g, e}. It can be verified that K is a W_Δ-K set.

However, its complement, K^c = {h, f}, is not I_Δ-perfect, since Γ(K^c) ≠ K^c. Therefore, this example confirms that the converse of Theorem 3.14 is not generally valid.

Theorems 3.17. Let (X, τ, I) be an ITS and let K ⊆ X. Then K^c is I_Δ-dense if and only if W_Δ(K) = ∅.

Proof. Suppose (X, τ, I) is an ITS and let K ⊆ X.

We observe the following equivalence: W_Δ(K)=∅⇔X\Γ(K^c)=∅⇔Γ(K^c)=X⇔K^c is I_Δ-dense. Hence, the result follows directly.

Remark 3.18: In general, the concepts of I_Δ-density and W_Δ-K sets are not mutually inclusive. Specifically, a set being I_Δ-dense does not guarantee that it is a W_Δ-K set and vice versa [22].

Example 3.19: Consider the ITS (X, τ, I), where X = {h, g, f, e}, τ = {∅, {h}, {f}, {h, g}, {h, f}, {h, g, f}, X} and I = {∅}.

In this context: The empty set ∅ is a W_Δ-K set since ∅⊆cl(W_Δ(∅))=cl(∅)=∅. However, it is clearly not I_Δ-dense because its complement X does not satisfy X = Γ(X). Conversely, the singleton set {f} is I_Δ-dense, as its complement satisfies X=Γ({f}). However, it fails to satisfy the condition {f}⊆cl(W_Δ({f})), and is therefore not a W_Δ-K set.

Remark 3.20: In the framework of ideal topological spaces, the notions of I_Δ-density and W_Δ-K sets are generally independent of one another. That is to say, a set being I_Δ-dense does not necessarily imply that it is a W_Δ-K set; conversely, a W_Δ-K set may not be I_Δ-dense.

Example 3.21: Let X = {h, g, f, e} and τ = {∅, {h}, {e}, {h, g}, {h, e}, {h, g, e}, X}.

and $I = \{\emptyset\}$. Consider the ideal topological space (X, τ, I).The empty set ∅ is trivially a $W_\Delta - K$ set, as it is contained in the closure of its own W_Δ -image. However, it fails to be I_Δ-dense because its complement X does not satisfy the condition $\Gamma(X) = X$. On the other hand, the singleton set $\{\not f\}$ is I_Δ-dense, since the interior closure under Δ covers the entire space. Nonetheless, it does not satisfy the conditions for being a $W_\Delta - K$ set, as $\{\not f\} \nsubseteq cl(W_\Delta(\{\not f\}))$.

Theorems 3.22. Let (X, τ, I) be an ITS such that cl(τ)∩I = {∅}. In this case, the empty set is the only subset of I that qualifies as a W_Δ-K set.

Proof. Consider (X, τ, I) with the condition that cl(τ)∩I = {∅}. If K is in I, then by Theorem 2.3, W_Δ(K) = ∅ and therefore cl(W_Δ(K)) = ∅. Now suppose that K is a W_Δ-K set. By definition 3.2, then, K⊆cl(W_Δ(K))=∅, which implies K=∅. Therefore, under the given condition, the only W_Δ-K set in the ideal I is the empty set.

Remark 3.23: Within the framework of ITS, it is important to note that being Δ-dense in itself does not necessarily imply being a W_Δ-K set. Conversely, a W_Δ-K set may fail to be Δ-dense in itself. This distinction illustrates the non-equivalence between these two classes of sets.

Example 3.24: Consider the ITS X = {h, g, f, e} with the topology τ = {∅, {h}, {e}, {h, f}, {h, e}, {h, f, e}, X} and the ideal I = P(X), the power set of X.

The singleton set K = {h} is a W_Δ-K set; however, it does not satisfy the condition of being Δ-dense in itself.

Example 3.25: Let X = {h, g, f, e} and let the topology be τ = {∅, {h}, {e}, {h, f}, {h, e}, {h, f, e}, X}. In this case, the set K = {h} is Δ-dense in itself, yet it is not a W_Δ-K set.

Theorems 3.26. Let (X, τ, I) be an ITS such that cl(τ)∩I = {∅}. Then, every W_Δ-K set in X is Δ-dense in itself.

Proof. Let (X, τ, I) be an ITS with cl(τ)∩I = {∅} and let K ⊆ X be a W_Δ-K set. By definition, K⊆cl(W_Δ(K)). Since cl(τ)∩I = {∅}, Theorem 2.3 implies that W_Δ(K) ⊆ Γ(K). Therefore, K⊆cl(W_Δ(K))⊆cl(Γ(K)). Since ℛ(K) is closed, we conclude that K⊆ℛ(K), which implies that K is Δ-dense in itself.

Remark 3.27. In an ITS, being a $W_\Delta - K$ set does not necessarily imply L_Δ-perfectness. That is, a set may satisfy the conditions of being $W_\Delta - K$ without being L_Δ-perfect.

Example 3.28: Consider the ITS (R, P(X), I_f), where P(X) denotes the power set topology on R and I_f is the ideal of all finite subsets of R.

In this space, the entire set R is a W_Δ-K set. However, it is not L_Δ-perfect.

Theorem 3.29: Let (X, τ, I) be an ITS and let K ⊆ X. If K is semi-θ-open, then K is a W_Δ-K set.

Proof. Assume that K⊆X is semi-θ-open in the ITS (X, τ, I). This implies that K⊆cl_θ(int_θ(K)). Now suppose that x is in X and x is not in the closure of W_Δ(K). Then there exists an open set G in τ containing x such that G∩W_Δ(K)=∅. Therefore, x belongs to G, which belongs to X\W_Δ(K) = Γ(K^c). Since Γ(K^c) is closed, x must be in the closure of G, which is contained in Γ(K^c). This implies that x is in the interior of Γ(K^c). Also, since Γ(K^c) ⊆ cl_θ(K^c), it follows that int_θ(Γ(K^c)) ⊆ int_θ(cl_θ(K^c)). Therefore, x belongs to X \ int_θ (cl_θ (K^c)) = cl_θ (int_θ (K)). Since A⊆cl(int(K)), it follows that x∉K. Therefore, by the contrapositive, we conclude that K ⊆ cl(W_Δ(K)), which implies that K is a W_Δ-K set.

Remark 3.30: In an ITS, the converse of Theorem 3.29 does not generally hold; that is to say, a W_Δ-K set is not necessarily semi-θ-open.

Example 3.31. Consider the ITS (X, τ, I), where X = {h, g, f, e} and τ = {∅, {h}, {e}, {h, g}, {h, e}, {h, g, e}, X} and I = {∅, {h}}. . In this space, the subset K = {g, f} is a W_Δ-K set, but it is not semi-θ-open.

Theorems 3.32. Let (X, τ, I) be an ITS and suppose K ∊ I. If K is a W_Δ-K set, then every subset H⊆K is also a W_Δ-K set.

Proof. Assume that K is a W_Δ-K set in an ITS (X, τ, I) and that K belongs to I. Let H⊆K. Since I is hereditary, it follows that H⊆I. We therefore have: Γ(X) = Γ(H^c) = Γ(K^c). This implies that cl(W_Δ(K)) = cl(W_Δ(H)). Since K⊆cl(W_Δ(K)), it also follows that H⊆K⊆cl(W_Δ(K))=cl(W_Δ(H)). Therefore, H is a W_Δ-K set**.**Proof. Assume $\mathcal{K} \in I$ and that $\mathcal{K}$ is a $W_\Delta - K$ set in an ITS (X, τ, I) . Let $\mathcal{H} \subseteq \mathcal{K}$. Since I is hereditary, it follows that $\mathcal{H} \in I$, we have: $\Gamma(X) = \Gamma(\mathcal{H}^c) = \Gamma(\mathcal{K}^c)$. This implies $cl(W_\Delta(\mathcal{K})) = cl(W_\Delta(\mathcal{H}))$. Given that $\mathcal{K} \subseteq cl(W_\Delta(\mathcal{K}))$, we also have

$\mathcal{H} \subseteq \mathcal{K} \subseteq cl(W_\Delta(\mathcal{K})) = cl(W_\Delta(\mathcal{H}))$. Therefore, $\mathcal{H}$ is a $W_\Delta - K$ set.

Remark 3.33. In an ITS, the intersection of two $W_\Delta - K$ sets do not necessarily yield a $W_\Delta - K$ set.

Example 3.34: Let K = {h, g, f, e}, T = {∅, {h}, {e}, {h, g}, {h, e}, {h, g, e}, X} and I = {∅, {a}}. Consider the subsets K = {g, f} and H = {f, e}. Both K and H are W_Δ-K sets in this structure. However, their intersection, K∩H = {f}, is not a W_Δ-K set.

Theorem 3.35: Let (X, τ, I) be an ITS and let K ⊆ X be a non-empty set. If, for all x in K, there exists an open set G in τ(x) such that cl(G) \ K^c in I, then K^c is a W_Δ-K set.

Proof. Assume that (X, τ, I) is an ITS and that K ⊆ X is a non-empty set. Suppose that, for each x in K, there exists an open set G in τ(x) such that cl(G) \ K^c in I. This condition implies that cl(G) \ ∅ K^c in I, meaning x in / G(K^c). Therefore, x is in X \ Γ(K^c), which is equal to W_Δ(K). Thus, K is in W_Δ(K), which is in turn contained within cl(W_Δ(K)). Therefore, K⊆cl(W_Δ(K)) and so K is a W_Δ-K set.

Remark 3.36: The converse of Theorem 3.35 does not necessarily hold. That is to say, a set being W_Δ-K does not imply the existence of an open neighbourhood G in the topology of K for each x in K such that the closure of G minus K is in I.

Example 3.37: Consider the set X = {h, g, f, e}, the topology τ = {∅, {h}, {e}, {h, g}, {h, e}, {h, g, e}, X} and the ideal I = {∅}. In an ITS (X, τ, I), the set K = {f, e} is a W_Δ-K set. However, for any element of K, the only open neighbourhood is X∉τ(f) and cl(X)\K^c=X\

4. Conclusion

The examination of W_Δ several significant findings are revealed while considering -K sets in the context of perfect topological spaces. Its structural flexibility is evident in the fact that the class of $W_\Delta - K$ sets is larger than many traditional generalized open sets, including θ-open, ω-open, as well as θ-semiopen sets. One important finding indicates that a set A must be a $W_\Delta - K$ set if its complement is I_Δ-perfect. The opposite is typically true, though, emphasizing that ideal completeness in the complement is not guaranteed by the $W_\Delta - K$ characteristic alone. Furthermore, not all either open or closed sets have this attribute, demonstrating a true containment connection rather than equivalence, even if every θ-open set is a $W_\Delta - K$ set.

Reiterating the complex and nontrivial relationships among ideal-theoretic operators and expanded topological notions, the counterexamples presented highlight the non-reversibility of various consequences. Finally, the compilation of W_Δ In supra-topological research, K sets emerge as a fundamental structure that unifies several generalized openness ideas and provides fresh insights into closure behaviors associated to ideals.

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