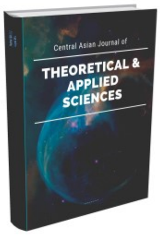

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Article

δ_σ *–Continuous Function in Proximity Topological Spaces*

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Abstract: We have composed this article to distinguish a new sort of continuity in the generated topology in the space of proximity, which depends on opening group in proximity space, and we have presented some of its identifications and types among it and other sorts, which we have shown in a prior research.

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INTRODUCTION

Because of development in many different types of sciences, technical problem, many scientific, and engineering, this problem need fast solution and non-traditional to reduce the confusion in these types pure and sciences applied [5,12]. One between important notions that play unfiled role in solving many problem is the concept nearness (proximity) which has an important and basic role in solving various problem in the fields of image analysis, face recognition, image processing and information systems [2,5,6].
Proximity introduce in 1909 by Reize and in 1952 Efromovic developed [1,9,10] it as well as 1906 [2,4], the concept of continuity is considering one of essential and fundamental concept for topological propositions, as it has been studied in detail and through different types, both notions are firstly mentioned by Fr`echet, topological structure [2,4], Kuratowski, in 1933 [2,4,9].
Therefore, in a previous study, we have shown a sort of continuity that we called δ_ω-continuous and in this research, we will show a new type of continuity, that we called named δ_σ-continuous, and we will show through the research the qualification of the new sort and the relationship between it δ- continuous, δ_ω-continuous.

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In that part, we review the most important acquaintances and theorem that have been studied in proximity space and necessary in our work, through which we arrive at the most important properties of the new type of continuity, we studied in this article.

Definition 1 [5]

δ is relation on the family $\mathcal{P}(\mathfrak{X})$ of all subsets of a set $\mathfrak{X}$ called a proximity on $\mathfrak{X}$ if the following conditions are satisfies:

(P1) If $\mathcal{M}\delta\,\mathcal{B}$, then $\mathcal{B}\;\delta\;\mathcal{M}$;

(P2) $\mathcal{M}\;\delta\;(\mathcal{B}\;\cup\;\mathcal{C})$ If and only if either $\mathcal{M}\delta\,\mathcal{B}$ or $\mathcal{M}\;\delta\;\mathcal{C}$;

(P3) $\mathfrak{X}\overline{\delta}\,\emptyset$;

(P4) $\{x\}\;\delta\;\{x\}$ For each $x \in \mathfrak{X}$;

(P5) If $\mathcal{M}\bar{\delta}\,\mathcal{B}$, then there exist $E \in \mathrm{P}(\mathfrak{X})$ such that $\mathcal{M}\bar{\delta}\,E$ and $X - E\bar{\delta}\,\mathcal{B}$.
The pair $(\mathfrak{X},\delta)$ called a proximity space. If (P4) is replaced by $(\mathrm{P}'4)\{x\}\delta\{y\}$ if and only $if\ x = y$, then δ is called a separated proximity relation and $(\mathfrak{X},\delta)$ is called a separated proximity space.

Example 1 [5]:

We have discrete δ_D proximity space and indiscrete δ_I proximity space of any subset $\mathcal{M},\mathcal{B}$ of power set $\mathcal{P}(\mathfrak{X})$ of a set $\mathfrak{X}$ defined as:
$\mathcal{M}\delta_D\mathcal{B}$ if and only if $\mathcal{M} \cap \mathcal{B} \neq \emptyset$, then δ_D is discrete proximity on a set $\mathfrak{X}$.
If $\mathcal{M}\delta_I\mathcal{B}$ for every pair of non-empty subset $\mathcal{M}$ and $\mathcal{B}$ of $\mathfrak{X}$, then the obtain the indiscrete proximity on $\mathfrak{X}$.

Definition 2 [5]:

If δ_1 and δ_2 are two elements of class P of all proximities defined on a set $\mathfrak{X}$, can be partially order by inclusion we define $\delta_1 > \delta_2$ if and only if $\mathcal{M}\ \delta_1\mathcal{B}$ implies $\mathcal{M}\ \delta_2\mathcal{B}$ .In this case we say that δ_1 is finer than δ_2, or δ_2 is coarser than δ_1.

Proposition 1 [4, 5, 8]:

Let $(\mathfrak{X},\delta)$ be a proximity space. Then
(a) if $\mathcal{M}\ \delta\ \mathcal{B}$ and $\mathcal{B} \subseteq C$, then $\mathcal{M}\ \delta\ C$;
(b) if $\mathcal{M}\ \bar{\delta}\mathcal{B}$ and $C \subseteq \mathcal{B}$, then $\mathcal{M}\bar{\delta}C$;
(c) if there exists a point $x \in \mathfrak{X}$ such that $\mathcal{M}\delta\ \{x\}$ and $\{x\}\ \delta\ \mathcal{B}$, then $\mathcal{M}\ \delta\ \mathcal{B}$;
(d) if $\mathcal{M} \cap \mathcal{B} \neq \emptyset$, then $\mathcal{M}\ \delta\ \mathcal{B}$;
(e) $\mathcal{M}\bar{\delta}\emptyset$ for every $\mathcal{M} \subseteq \mathfrak{X}$;
(f) if $\mathcal{M}\ \delta\ \mathcal{B}$, then $\mathcal{M} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$.

Proposition 2:

The axiom (P5 in Definition 1] is equivalent to any of the following statements:
If $\mathcal{M}\bar{\delta}\mathcal{B}$, then there are sets $\mathcal{W}$ and $N, \mathcal{W} \cup N = \mathfrak{X}$ such that $\mathcal{M}\bar{\delta}\mathcal{W}, \mathcal{B}\bar{\delta}N$;
If $\mathcal{M}\bar{\delta}\mathcal{B}$, then there are sets $\mathcal{W}$ and N such that $\mathcal{M}\bar{\delta}\mathfrak{X} - \mathcal{W}, \mathfrak{X} - N\bar{\delta}\mathcal{B}$ and $\mathcal{W}\bar{\delta}N$;
If $\mathcal{M}\bar{\delta}\mathcal{B}$, then there are two sets H and $\mathcal{W}$ such that $\mathcal{M}\bar{\delta}\mathfrak{X} - \mathcal{W}$,and $\mathcal{B}\bar{\delta}\mathfrak{X} - N, \mathcal{W} \cap N = \emptyset$.

Definition 3 [2,5]:

Let (X,δ) be a proximity space, then for all $\mathcal{M},\mathcal{B} \subset \mathfrak{X}$, $\mathcal{B}$ a proximity or δ −neighborhood of $\mathcal{M}$ and denoted that relation as $\mathcal{M} \ll \mathcal{B}$ if and only if
$\mathcal{M}\bar{\delta}\ \mathfrak{X} - \mathcal{B}$.

Theorem 1 [2]:

Let $(\mathfrak{X},\delta)$ be a proximity space. Then the relation $\ll$ satisfies the following properties:
$(O1)$ $\mathfrak{X} \ll \mathfrak{X}$;
$(O2)$ $If\ \mathcal{M} \ll \mathcal{B}$, then $\mathcal{M} \subset \mathcal{B}$;
$(O3)$ $\mathcal{M} \subset \mathcal{B} \ll C \subset D$ implies $\mathcal{M} \ll D$;
$(O4)$ $\mathcal{M} \ll \mathcal{B}$ implies $\mathfrak{X} - \mathcal{B} \ll \mathfrak{X} - \mathcal{M}$;
$(O5)$ $\mathcal{M} \ll \mathcal{B}_k$ is true for k = 1, 2, …. , n if and only if $\mathcal{M} \ll \cap_{k=1}^{n} \mathcal{B}_k$;
$(O6)$ If $\mathcal{M} \ll \mathcal{B}$, then there exists a set $N \subset X$ such that $\mathcal{M} \ll N \ll \mathcal{B}$. This is implies $\mathcal{M} \ll int\ N \subset cl\ N \ll int\mathcal{B} \subset cl\mathcal{B}$. If δ is a separated proximity , then
$(O7)$ $\{x\} \ll \mathfrak{X} - \{y\}$ if and only if $x \neq y$.

Corollary 1[5,7]:

If $\mathcal{M}_k \ll \mathcal{B}_k$ k = 1, 2, … n, then

$$\bigcap_{k=1}^{n} \mathcal{M}_k \ll \bigcap_{k=1}^{n} \mathcal{B}_k \quad and \quad \bigcup_{k=1}^{n} \mathcal{M}_k \ll \bigcup_{k=1}^{n} \mathcal{B}_k$$

Remark 1 [5]:

The family $\mathcal{F}(\mathcal{M})$ of all δ-neighborhoods of a set $\mathcal{M}$ in proximity space $(\mathfrak{X},\delta)$ and δ-neighborhoods in general is not open set with respect to this topology.

Proposition 3 [5]:

Let $(\mathfrak{X},\delta)$ be a proximity space. Then
(a) $\mathcal{B} \in \mathcal{F}(\mathcal{M})$ implies $\mathcal{M} \subset \mathcal{B}$;
(b) $\mathcal{B} \in \mathcal{F}(\mathcal{M})$ implies $\mathfrak{X} - \mathcal{M} \in \mathcal{F}(\mathfrak{X} - \mathcal{B})$;
(c) If $\mathcal{M} \subset \mathcal{B}$, then $\mathcal{F}(\mathcal{M}) \subset \mathcal{F}(\mathcal{B})$);
(d) $\mathcal{F}(\mathcal{M} \cup \mathcal{B}) = \mathcal{F}(\mathcal{M}) \cap \mathcal{F}(\mathcal{B})$;
(e) If $\mathcal{B} \in \mathcal{F}(\mathcal{M})$), then there exists $C \in \mathcal{F}(\mathcal{M})$ such that $\mathcal{B} \in \mathcal{F}(C)$);

(f) $\mathcal{F}(\mathcal{M}) \cap \mathcal{F}(\mathcal{B})) \subset \mathcal{F}(\mathcal{M} \cap \mathcal{B})$, where $\mathcal{F}(\mathcal{M}) \cap \mathcal{F}(\mathcal{B}) = \{C \cap D : C \in \mathcal{F}(\mathcal{M}), D \in \mathcal{F}(\mathcal{B})\}$.

TOPOLOGY GENERATED BY A PROXIMITY

In this part we shall be consider the topology on $\mathfrak{X}$ induced by a Proximity on $\mathfrak{X}$, and studies some definition and elementary properties.

Theorem 2 [2, 8, 9]:

If $(\mathfrak{X}, \delta)$is a proximity space, then the family T_δ is a topology on the set $\mathfrak{X}$.

Definition 2.2[6,7]:

Let $(\mathfrak{X}, \delta)$ be a proximity space. A subset $F \subset \mathfrak{X}$ is defined to be closed if and only if $x\,\delta\,F$ implies $x \in F$. By T_δ denote the family of complements of all the sets defined in such a way.

Proposition 4[6]:

The topology T_δ generated by proximity relation on space X is regular.

Proposition 5 [6]:

For any two elements δ_1, δ_2 of $\mathcal{P}$, $\delta_1 > \delta_2$ if and only if $\mathfrak{Y}\delta_1\mathcal{Q}$ implies $\mathcal{Q}\delta_2\mathfrak{Y}$.

Definition 4[7]:

If $\mathfrak{S}$is a subset of a proximity space $(\mathfrak{X}, \delta)$, then $\mathfrak{S}$is open in topology T_δ if and only if $\{x\}\,\bar{\delta}\,\mathfrak{X} - \mathfrak{S}$ for every $x \in \mathfrak{S}$.

Proposition 6 [6,7]:

For any two proximity relations δ_1, δ_2 in $\mathfrak{X}$, if $\delta_1 < \delta_2$, then $\mathrm{T}_{\delta_1} \subset \mathrm{T}_{\delta_2}$.

Proposition 7 [10]:

If $\mathcal{M}$ and $\mathcal{B}$ are subsets of a proximity space $(\mathfrak{X}, \delta)$, then $\mathcal{M}\;\bar{\delta}\;\mathcal{B}$ implies:

$\overline{\mathcal{B}} \subset \mathfrak{X} - \mathcal{M}$; and (ii) $\mathcal{B} \subset int\,(\mathfrak{X} - \mathcal{M})$.

Proposition 8 [4,10]:

If $\overline{\mathcal{M}}$ and $Int\mathcal{M}$ detnote, respectively, the closure and the interior of the set $\mathcal{M}$ of a proximity space $(\mathfrak{X}, \delta)$ with respect to the topology T_δ, then

$\mathcal{M} \ll \mathcal{B}$ implies $\overline{\mathcal{M}} \ll \mathcal{B}$ (ii) $\mathcal{M} \ll \mathcal{B}$ implies $\mathcal{M} \ll Int\,\mathcal{B}$.

Theorem 3[5]:

Let $\emptyset \neq \mathcal{G} \subset \mathfrak{X}\; and\; (\mathfrak{X}, \delta)$ be a proximity space, for $\mathcal{M}, \mathcal{B} \subset \mathcal{G}$ let $\mathcal{M}\delta_{\mathcal{G}}\mathcal{B}$ if and only if $\mathcal{M}\delta\,\mathcal{B}$.Then (Y, δ_Y) is a proximity space.

Proposition 9 [5]:

Let $(\mathfrak{X}, \delta)$ be a proximity space, $\emptyset \neq \mathcal{G} \subset \mathfrak{X}$, then $\mathcal{F}_{\mathcal{G}}(\mathcal{M}) = \{\mathcal{B} \subseteq \mathcal{G}, \mathcal{M} \ll_{\mathcal{G}} \mathcal{B}\} = \mathcal{F}_{\mathfrak{X}}(\mathcal{M}) \cap \{Y\}$.

Definition 5 [5]:

Let $\emptyset \neq \mathcal{G} \subset \mathfrak{X}$, $(\mathfrak{X}, \delta)$ be a proximity space, The ordered pair $(\mathcal{G}, \delta_{\mathcal{G}})$ is called the proximity subspace of the proximity space $(\mathfrak{X}, \delta)$ such that the proximity relation $\delta_{\mathcal{G}}$ defined on the subset γ of the set $\mathfrak{X}$ is called the restriction on $\gamma\; of$ the proximity δ and is denoted by $\delta_{\mathcal{G}}$.

PROXIMALLY CONTINUOUS FUNCTION.

In this part, we will discuss the most important definition and special feature of continuity in proximity space.

Definition 6 [2,3,4]:

Let $(\mathfrak{X}, \delta_\chi)$ and $(\mathcal{G}, \delta_{\mathcal{G}})$ be two proximity spaces. The mapping $f : \mathfrak{X} \to \mathcal{G}$ is said to be proximally or δ-continuous if $\mathcal{M}\delta_{\mathfrak{X}}\,\mathcal{B}$ implies $f(\mathcal{M})\delta_{\mathcal{G}}\,f(\mathcal{B})$ for every two sets $\mathcal{M}, \mathcal{B} \subset \mathfrak{X}$.

Proposition 10[6]:

Let us consider a mapping $\mathrm{f} : \mathfrak{X} \longrightarrow \mathcal{G}$, where $(\mathcal{G}, \delta)$ is a proximity space and let us define a relation on the power set $P(\mathfrak{X})$ of the set $\mathfrak{X}$ in the following way:

$\mathcal{M}\delta^*\,\mathcal{B}$ if and only if $\mathrm{f}\,(\mathcal{M})\;\delta\;\mathrm{f}\,(\mathcal{B})$

The proximity relation δ^* defined in such a way is called the inverse image of the proximity δ and denoted by $f^{-1}(\delta)$.

Proposition 11 [2,5]:

If $f : \mathfrak{X} \to \mathcal{G}$ and $\delta\; is$ a proximity on the set $\mathcal{G}$, then $f^{-1}(\mathrm{T}_\delta) = \mathrm{T}(f^{-1}(\delta))$.

Corollary 2 [6]:

If $f : \mathfrak{X} \to \mathcal{G}$ and if δ_1 and δ_2 are the $\mathfrak{X}$ proximity on $\mathcal{G}$ for which $\delta_1 < \delta_2$, then $f^{-1}(\delta_1) < f^{-1}(\delta_2)$ hold.

Proposition 12 [2, 3]:

A mapping $f : \mathfrak{X} \to \gamma$ of a proximity space$(\mathfrak{X}, \delta_{\mathfrak{X}})$ into a proximity space $(\mathcal{G}, \delta_{\mathcal{G}})$ is δ-continuous if and only if for every two sets , $K \subset \mathcal{G}$, $R\bar{\delta}_{\mathcal{G}}\, K$ implies f^{-1}(R) $\bar{\delta}_{\mathfrak{X}} f^{-1}$(K).

Corollary 3 [6]:

Let $f : \mathfrak{X} \to \mathcal{G}$ be a mapping from a set $\mathfrak{X}$ on a proximity space (γ, δ_γ), then $\delta_{\mathfrak{X}} = f^{-1}(\delta_\gamma)$) is the coarsest proximity on $\mathfrak{X}$ for which f is a δ-coontinuous mapping.

δ_ω −CONTINUOUS FUNCTION.

Definition 7[1]:

Let $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ be a mapping, the f is said to be weakly continuous function in proximity space iff for all $x \in \mathfrak{X}$, and for all $\mathcal{H} \subset \mathcal{G}$, $f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - \mathcal{H}$, there exists $\mathcal{U} \subset \mathfrak{X}$, $x\bar{\delta}_{\mathfrak{X}}\mathfrak{X} - \mathcal{U}$, $f(\mathcal{U})\bar{\delta}_{\mathcal{G}}\mathcal{G} - \mathcal{H}$ and denoted δ_ω −continuous.

Proposition 13[1]:

$f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ is δ_ω −continuous iff for all $x \in \mathfrak{X}$ and for all $\mathcal{H} \subset \mathcal{G}, f(x) \ll \mathcal{H}$, there exists $U \subset \mathfrak{X}, x \ll U, f(U) \ll \mathcal{H}$.

Proposition 14[1]:

If $f: (\mathfrak{X}, \delta_{I\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$, then f is not δ_ω −continuous function.

Proposition 15[1]:

If $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ is δ_ω −continuous function, then f is continuous.

Proposition 16[1]:

If $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ is δ_ω −continuous function, then does not necessary f is δ −continuous function.

Proposition 17[1]:

If $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ bijective δ −continuous function, then f is δ_ω −continuous function.

δ_σ −CONTINUOUS FUNCTION

Definition 8:

Let $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ be any function, then f is called almost continuous function in proximity space and it denoted simply (δ_σ −continuous function in proximity space) if and only if for all $x \in \mathfrak{X}, V \subset \mathcal{G}$ such that $f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - V$, there exists $\mathcal{U} \in \mathrm{T}_{\delta_{\mathfrak{X}}}$ such that $f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - \bar{V}$ for all $x \in \mathcal{U}$ and $f(\mathcal{U}) \subset \bar{V}$.

Example 2:

Let $(\mathfrak{X}, \mathrm{T})$ is normal space ($\mathfrak{H}\bar{\delta}_{\mathfrak{X}}\mathfrak{B} \leftrightarrow \bar{\mathfrak{H}} \cap \bar{\mathfrak{B}} = \emptyset$), $\delta_{\mathfrak{X}}$ is proximity relation on $\mathfrak{X}$ and $(\mathcal{G}, \delta_{\mathcal{G}})$ is proximity space such that $\delta_{D\mathcal{G}}$ is discrete proximity space, then $f: (\mathfrak{X}, \mathrm{T}_{\delta_{\mathfrak{X}}}) \to (\mathcal{G}, \delta_{D\mathcal{G}})$ is δ_σ −continuous function, $\mathfrak{X} = \{1, 2, 3\}$, $\mathcal{G} = \{a, \mathfrak{h}\}$,

$\mathrm{T}_{\delta_{\mathfrak{X}}} = \{\emptyset, \mathfrak{X}, \{1\}, \{2,3\}\}$ is proximity topological space, $f(1) = a, f(2) = f(3) = \mathfrak{h}$.

Solution

Let $x = 1$,

If $V = \{a\}$, $f(1) = a\bar{\delta}_{\mathcal{G}}\mathcal{G} - \{a\} = \{\mathfrak{h}\}$, there exists

$\mathcal{U} = \{1\} \in \mathrm{T}_{\delta_{\mathfrak{X}}}$ and $f(1) = \{a\}\bar{\delta}_{\mathcal{G}}\mathcal{G} - \overline{\{a\}} = \{\mathfrak{h}\}$, and $f(\{1\}) = \{a\} \subset \overline{\{a\}}$.

If $V = \{\mathfrak{h}\}$, $f(1) = a\delta_{\mathcal{G}}\mathcal{G} - \{\mathfrak{h}\} = \{a\}$,

Let $x = 2$,

If $V = \{a\}$, $f(2) = \mathfrak{h}\delta_{\mathcal{G}}\mathcal{G} - \{a\} = \{\mathfrak{h}\}$,

If $V = \{\mathfrak{h}\}$, $f(2) = \mathfrak{h}\bar{\delta}_{\mathcal{G}}\mathcal{G} - \overline{\{\mathfrak{h}\}} = \{a\}$, and $f(2) = \mathfrak{h} \subset \overline{\{\mathfrak{h}\}}$there exists

$$\mathcal{U} = \{2,3\} \in \mathrm{T}_{\delta_{\mathfrak{X}}} \text{ and } \begin{cases} f(2) = \mathfrak{h}\ \bar{\delta}_{\mathcal{G}}\mathcal{G} - \overline{\{\mathfrak{h}\}} = \{a\}, and\ f(2) = \mathfrak{h} \subset \overline{\{\mathfrak{h}\}} \\ f(3) = \mathfrak{h}\ \bar{\delta}_{\mathcal{G}}\mathcal{G} - \overline{\{\mathfrak{h}\}} = \{a\}\ and\ f(3) = \mathfrak{h} \subset \overline{\{\mathfrak{h}\}} \end{cases}$$

Let $x = 3$,

If $V = \{a\}$, $f(3) = \mathfrak{h}\delta_{\mathcal{G}}\mathcal{G} - \{a\} = \{\mathfrak{h}\}$,

If $V = \{\mathfrak{h}\}$, $f(3) = \mathfrak{h}\bar{\delta}_{\mathcal{G}}\mathcal{G} - \{\mathfrak{h}\} = \{a\}$, $and\ f(3) = \mathfrak{h} \subset \overline{\{\mathfrak{h}\}}$, there exists $\mathcal{U} = \{2,3\}$ and $\begin{cases} f(2) = \mathfrak{h}\ \bar{\delta}_{\mathcal{G}}\mathcal{G} - \overline{\{\mathfrak{h}\}} = \{a\}, and\ f(2) = \mathfrak{h} \subset \overline{\{\mathfrak{h}\}} \\ f(3) = \mathfrak{h}\ \bar{\delta}_{\mathcal{G}}\mathcal{G} - \overline{\{\mathfrak{h}\}} = \{a\}, and\ f(3) = \mathfrak{h} \subset \overline{\{\mathfrak{h}\}} \end{cases}$

Thus, f is δ_σ −continuous function.

Proposition 18:

Not all δ_σ −continuous function is δ −continuous function and defined as $f: (\mathfrak{X}, \mathrm{T}_{\delta_{\mathfrak{X}}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$, $\mathfrak{X} = \{1,2\}$, $\mathcal{G} = \{\mathfrak{h}, a\}$, $f(1) = a$, $f(2) = \mathfrak{h}$ and defined $\delta_{\mathfrak{X}}$ as $\mathfrak{H}\delta_{\mathfrak{X}}\mathfrak{M} \leftrightarrow \mathfrak{H} \neq \emptyset, \mathfrak{M} \neq \emptyset$, we will find $\{a\}\bar{\delta}_{\mathcal{G}}\{\mathfrak{h}\}$ but
$f^{-1}(\{a\}) = \{1\}\delta_{\mathfrak{X}}f^{-1}(\{\mathfrak{h}\}) = \{2\}$, then f is not δ −continuous.

Remark 2:

Every bijective δ −continuous and continuous $f: (\mathfrak{X}, \mathrm{T}_{\delta_{\mathfrak{X}}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ is δ_σ −continuous.

Proof:

Let $x \in \mathfrak{X}, V \subset \mathcal{G}, f(\mathfrak{X})\bar{\delta}_{\mathcal{G}}\mathfrak{X} - V$, since f is δ −continuous, then $f^{-1}(f(x))\ \bar{\delta}_{\mathcal{G}} f^{-1}(\mathcal{G} - V)$ and ;hence, $x\bar{\delta}_{\mathcal{G}}\mathcal{G} - f^{-1}(V)$. Now let $f^{-1}(V) = \mathcal{U}$ and since f is continuous form that we yet there exists $U \in \mathrm{T}_{\delta_{\mathfrak{X}}}$, $\mathfrak{X} \in U$ and because $f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - V$, then [by Proposition 1.3.6 (b)] $f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - \bar{V}$; hence, f is δ_σ − continuous function.

Proposition 19:

$f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ is δ_σ −continuous if and only if for all $x \in \mathfrak{X}, V \subset \mathcal{G}$,
$f(x) \ll V$ there exists $\mathcal{U} \in \mathrm{T}_{\delta_{\mathfrak{X}}}$ such that $f(x) \ll \bar{V}$ for all $x \in \mathcal{U}$ and $f(\mathcal{U}) \subset \bar{V}$.

Proof:

Let $f: \mathfrak{X} \to (\mathcal{G}, \delta_{\mathcal{G}})$ is δ_σ −continuous, then for all $x \in \mathfrak{X}$, $V \subset \mathcal{G}$, $f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - V$, $f(x) \ll V$, there exists $\mathcal{U} \in \mathrm{T}_{\delta_{\mathfrak{X}}}$, such that $f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - \bar{V}$ and [by Definition (1.6)] $f(x) \ll V$ for all $x \in \mathcal{U}$, and $f(\mathcal{U}) \subset \bar{V}$.

Conversely, if for all $x \ni \mathfrak{X}, V \subset \mathcal{G}, f(x) \ll V$ imply $f(x)\ \bar{\delta}_{\mathcal{G}}\mathcal{G} - V$, there exists $\mathcal{U} \in \mathrm{T}_{\delta_{\mathfrak{X}}}$ such that $f(x) \ll \bar{V}$, then [by Definition (1.6)]$f(x)\ \bar{\delta}_{\mathcal{G}}\mathcal{G} - \bar{V}$ for all $x \in \mathcal{U}$. Now, since $f(x)\ \bar{\delta}_{\mathcal{G}}\mathcal{G} - \bar{V}$, $f(x) \cap \bar{V}^c = \emptyset$, and then $f(x) \subset \bar{V}$ for all $x \in \mathcal{U}$, since $\cup \{f(x): x \in \mathcal{U}\} \subset \bar{V}$; hence, $f(\mathcal{U}) \subset \bar{V}$ that mean f is δ_σ −continuous function.

Remark 3:

$f: (\mathfrak{X}, \mathrm{T}_{\delta_{\mathfrak{X}}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ is constant, then f is δ_σ −continuous function since
if $f: (\mathfrak{X}, \mathrm{T}_{\delta_{\mathfrak{X}}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ such that, $f(x) = a$ for all $x \in \mathfrak{X}$, a is constant and let
$x \in \mathfrak{X}, V \subset \mathcal{G}, f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - V$, then for all $\mathcal{U} \in \mathrm{T}_{\delta_{\mathfrak{X}}}$ and for all $x \in \mathcal{U}$, $f(\mathcal{U}) = a$, and since $V \subseteq \bar{V}, \mathcal{G} - \bar{V} \subset \mathcal{G} - V$ and [by Proposition 1.3.6 (b)] $f(x) = a\bar{\delta}_{\mathcal{G}}Y - \bar{V}$, $f(\mathcal{U}) \subset \bar{V}$, then f is δ_σ −continuous function.

Proposition 20:

Let $f: \mathfrak{X} \to (\mathcal{G}, \delta_{\mathcal{G}})$ and $\mathcal{G}$ is single set $\mathcal{G} = \{a\}$, then f is not δ_σ −continuous function.

Proof:

Since $\mathcal{G} = \{a\}$, then $\delta_\sigma = \{(\{a\}, \{a\})\}$ and since for all $x \in \mathfrak{X}$, there does not exists a proper set $V \subset \{a\} = \mathcal{G}$, such that $f(x) = a\bar{\delta}_{\mathcal{G}}\mathcal{G} - V$, then f is not δ_σ − continuous function.

Proposition 21:

Every proximity δ_ω − continuous function is δ_σ −continuous function and conversely is true.

Proof:

Let $x \in \mathfrak{X}$, $V \subset \mathcal{G}$, $f(x)\bar{\delta}_{\mathcal{G}}\mathcal{G} - V$, since $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ is proximity δ_ω −continuous function, then exists $\mathcal{U} \subset \mathfrak{X}$, $x\bar{\delta}_{\mathfrak{X}}\mathfrak{X} - \mathcal{U}$ and [Definition2.4],
$\mathcal{U} \in \mathrm{T}_{\delta_{\mathfrak{X}}}$ and $f(\mathcal{U})\bar{\delta}_{\mathcal{G}}\mathcal{G} - V$ and [by Theorem 1] $f(\mathcal{U}) \subset V$ and then, $f(\mathcal{U}) \subset \bar{V}$, that implies f is δ_σ −continuous function.

Remark 4:

Not every δ_σ −continuous function is δ_ω − continuous function since if we take
$f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$, $\mathfrak{X} = \{1,2\}$, $\mathcal{G} = \{a, \mathfrak{h}\}$ and $\delta_{\mathfrak{X}}$ is defined as $\mathfrak{H}\delta_{\mathfrak{X}}\ \mathfrak{M}$ if and only if $\mathfrak{H} \neq \emptyset, \mathfrak{M} \neq \emptyset$ and $\delta_{\mathcal{G}}$ defined as $\mathfrak{H}\delta_{\mathcal{G}}\mathfrak{M}$ if and only if $\mathfrak{H} \cap \mathfrak{M} \neq \emptyset$, and f defined as $f(1) = f(2) = a$, clearly f is δ_σ −continuous function f does not δ_ω − continuous function since if we take $x = 1$, $f(1) = a$ and let $V = \{a\}$ such that

$f(1)\bar{\delta}_{\mathcal{G}}\mathcal{G} - V = \mathcal{G} - \{a\} = \{\mathfrak{h}\}$ there exists only $\mathcal{U} = \mathfrak{X}$,$1\bar{\delta}_{\mathfrak{X}}\mathfrak{X} - \mathfrak{X} = \emptyset$ but $\mathfrak{X}$ does not proper set; hence, f is does not $\delta_\omega -$ continuous function.

Corollary 4:

The composition of two $\delta_{\sigma -}$continuous is $\delta_{\sigma -}$continuous.

Proof

Suppose that $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_{\mathcal{G}})$ is $\delta_{\sigma -}$continuous function , $\mathfrak{S}: (\mathcal{G}, \delta_{\mathcal{G}}) \to (\mathfrak{q}, \delta_{\mathfrak{q}})$ is $\delta_{\sigma -}$continuous functions too, to prove $(\mathfrak{S}of)(\mathbb{x})$ is $\delta_{\sigma -}$continuous, let for all $\mathbb{x} \in \mathfrak{X}$, $\mathfrak{W} \subset \mathfrak{q}, (\mathfrak{S}of)(\mathbb{x})\bar{\delta}_{\mathfrak{q}}\mathfrak{q} - \mathfrak{W}$, $\mathfrak{S}(f(\mathbb{x}))\bar{\delta}_{\mathfrak{q}}\mathfrak{q} - \mathfrak{W}$, and since $\mathfrak{S}$ is $\delta_{\sigma -}$continuous function and $f(\mathbb{x}) \in \mathcal{G}$, then, there exists $\mathfrak{E} \in T_{\delta_{\mathcal{G}}}$, $\mathfrak{S}(f(\mathbb{x}))\bar{\delta}_{\mathfrak{q}}\mathfrak{q} - \mathfrak{E}$, for all $f(\ \ \mathbb{x}) \in \mathfrak{E}, and\ f(\mathfrak{E}) \subset \overline{\mathfrak{W}}$, and since $\mathbb{x} \in \mathfrak{X}$, f $\delta_{\sigma -}$continuous, then for all $\mathfrak{E} \subset \mathcal{G}, f(\mathbb{x})\bar{\delta}_{\mathcal{G}}\mathcal{G} - \mathfrak{E}$, but $\mathfrak{E} \in T_{\delta_{\mathcal{G}}}$,$f(\mathbb{x})\bar{\delta}_{\mathcal{G}}\mathcal{G} - \overline{\mathfrak{E}}$, there exists $\mathcal{U} \in \mathfrak{X}, f(\mathbb{x})\ \bar{\delta}_{\mathcal{G}}\mathcal{G} - \overline{\mathfrak{E}}$ for all $\mathbb{x} \in \mathcal{U}$, and $f(\mathcal{U}) \subset \overline{\mathfrak{E}}$,then $(\mathfrak{S}of)(\mathbb{x})$ is $\delta_{\sigma -}$continuous function.

Corollary 5:

Let $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_1)$ is $\delta_{\sigma -}$continuous function, if δ_2 is another proximity relation defined on $\mathcal{G}$ coarser than δ_1, then $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_2)$ is $\delta_{\sigma -}$continuous function.

Proof:

Since $f: (\mathfrak{X}, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_1)$ is $\delta_{\sigma -}$continuous, then for all $x \in \mathfrak{X}, \mathcal{V} \subseteq \mathcal{G}$,
$f(x)\bar{\delta}_1\mathcal{G} - \mathcal{V}$, there exist $\mathcal{U} \in T_{\delta_{\mathfrak{X}}}$, $f(x)\bar{\delta}_1\mathcal{G} - \bar{\mathcal{V}}$ and $f(\mathcal{U}) \subset \bar{\mathcal{V}}$, for all $x \in \mathcal{U}$ and since $\delta_2 < \delta_1$ [by Definition 1.3], $f(x)\bar{\delta}_2\mathcal{G} - \mathcal{V}$ for all $x \in \mathcal{U}$, $f(\mathcal{U}) \subset \bar{\mathcal{V}}$ hence $f: (\mathfrak{X},, \delta_{\mathfrak{X}}) \to (\mathcal{G}, \delta_2)$ is $\delta_{\sigma -}$continuous function.

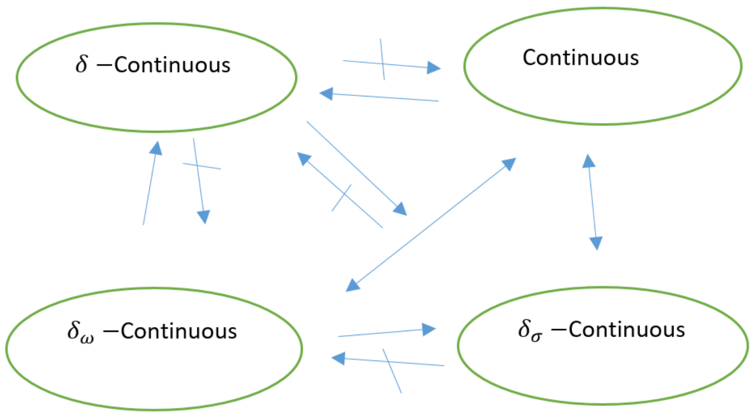
Proposition 22:

Every continuous function is $\delta_{\sigma -}$continuous.

Proof

Let $x \in X, V \subset Y, f(x)\bar{\delta}_Y Y - V$, then $f(x) \in V$; hence, [by Theorem 1] V is an open set, so because f is continuous, then $f^{-1}(V)$ is $T_{\delta_X} -$ open set, let
$U = f^{-1}(V)$; hence, $x \in f^{-1}(V)$ also $f(x)\bar{\delta}_Y Y - V$ for all $x \in U$. Now, since$f(U) \subset V$, then $f(U) \subset \bar{V}$; hence, f is $\delta_{\sigma -}$continuou

From above propositions we can draw diagram explain the relation between types of continuous in proximity space as follow:



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