



Article

On fuzzy topological K -algebras

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Abstract: This paper introduces and studies topological subalgebras with fuzzy edges and solutions in K -algebras, along with a number of their holdings. More specifically, our subject the characteristics of fuzzy topological ideals of the inverse and homomorphic images of K -algebras.

1. Introduction

An algebraic family known such as $K(G)$ -algebras or K -algebras of groups of G , which employ the resulting binary process, was introduced by K. H. Dar & M. Akram in [9]. They extended the scope of study for this algebra family in [10]. Fuzzy sets were first proposed by Zadeh [4]. Fuzzy set theory has since learned applications in a wide variety of domains. Evaluation regarding functional operations, topological spaces, groups, rings, near rings, vector spaces, and automation are mathematical concepts that numerous academics have confused. Fuzzy compactness, fuzzy $C5$ -disconnectness, and fuzzy Hausdorff spaces are among the characteristics studied in this article, along with fuzzy topological subalgebras and ideal in K -algebras. In meantime, we review the characteristics of fuzzy topological ideals of the inverse and homomorphic images of K -algebras. the twelfth 2. First Stages

Preliminaries

Definition 2.1. [9] Let (G, \cdot, e) be a group with the identity e such that $x^2 \neq e$ for some $x (\neq e) \in G$. A K -algebra is a structure $\mathcal{K} = (G, \cdot, \odot, e)$, where " \odot " is a binary operation on G which is induced from the operation " \cdot ", that satisfies the following:

$$(k1) (\forall a, x, y \in G) ((a \odot x) \odot (a \odot y) = (a \odot (y^{-1} \odot x^{-1})) \odot a),$$

$$(k2) (\forall a, x \in G) (a \odot (a \odot x) = (a \odot x^{-1}) \odot a),$$

$$(k3) (\forall a \in G) (a \odot a = e),$$

$$(k4) (\forall a \in G) (a \odot e = a),$$

$$(k5) (\forall a \in G) (e \odot a = a^{-1}).$$

If G is abelian, then conditions (k1) and (k2) can be written as follows:

$$(k1') (\forall a, x, y \in G) ((a \odot x) \odot (a \odot y) = y \odot x),$$

$$(k2') (\forall a, x \in G) (a \odot (a \odot x) = x),$$

respectively.

Definition 2.2. [9] A nonempty subset H of a K -algebra \mathcal{K} is called a *subalgebra* of \mathcal{K} if it satisfies:

$$(i) e \in H,$$

$$(ii) (\forall a, b \in H) (a \odot b \in H).$$

Definition 2.3. Let $\mathcal{K}_1 = (G_1, \cdot, \odot, e_1)$ and $\mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ be K -algebras. A mapping $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ of K -algebras is called a *homomorphism* if $f(x \odot y) = f(x) \odot f(y)$, $\forall x, y \in \mathcal{K}_1$.

Note that if f is a homomorphism, then $f(e_1) = e_2$.

Definition 2.4. A nonempty subset A of a K -algebra \mathcal{K} is called an *ideal* of \mathcal{K} if,

$$(i) e \in A,$$

$$(ii) (\forall y \in \mathcal{K})(\forall x \in A) (x \odot y) \in A \text{ and } y \odot (y \odot x) \in A \Rightarrow x \in A).$$

Definition 2.5. [1] Let X be a non-empty set. A *fuzzy* (sub)set A in X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$.

Definition 2.6. [1] Let A and B be two fuzzy sets in X , then following operations are valid :

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- (a) $(\forall x \in X)(A \subseteq B \Rightarrow \mu_A(x) \leq \mu_B(x))$,
 - (b) $(\forall x \in X)(A = B \Rightarrow \mu_A(x) = \mu_B(x))$,
 - (c) $(\forall x \in X)(A \cap B = \min(\mu_A(x), \mu_B(x)))$,
 - (d) $(\forall x \in X)(A \cup B = \max(\mu_A(x), \mu_B(x)))$.

Definition 2.7. The fuzzy sets ϕ_X and 1_X in X are defined by $\phi_X = \{x \in X : \mu(x) = 0\}$ and $1_X = \{x \in X : \mu(x) = 1\}$ respectively.

Definition 2.8. A *fuzzy topology* on a set X is a family τ of fuzzy sets in X which satisfies the following conditions:

- (i) $\emptyset_X, 1_X \in \tau$,
- (ii) If $A_1, A_2 \in \tau$, then $A_1 \cap A_2 \in \tau$,
- (iii) If $A_i \in \tau$ for all $i \in I$, then $\cup_{i \in I} A_i \in \tau$.

The pair (X, τ) is called a *fuzzy topological space*(FTS) and members of τ are called *open fuzzy sets*(OFSSs), and the complement of fuzzy open sets are *closed fuzzy sets*(CFSs).

Definition 2.9. [5] Let f be a mapping from a set X into set Y .

- (a) Let B be a fuzzy set in Y with membership function μ_B . The inverse image of B , denoted by $f^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.
- (b) Let A be a fuzzy set in X with membership function μ_A . The image of A , is denoted by $f(A)$, is the fuzzy set in Y with membership function $\mu_{f(A)}$ such that

$$f_{\text{sup}}(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.10. Let $A (A_i)$ be a fuzzy set with membership function $\mu_A (\mu_{A_i})$ in X and B be a fuzzy set with membership function μ_B in Y . Let $f : X \rightarrow Y$ be a function. Then

- (a) If f is surjective, then $f(f^{-1}(\mu_B)) = \mu_B$.
- (b) $f(\emptyset_X) = \emptyset_Y$.
- (c) $f(1_X) = 1_Y$, if f is surjective.
- (d) $f^{-1}(1_Y) = 1_X$.
- (e) $f^{-1}(\emptyset_Y) = \emptyset_X$.
- (f) $f(\cup A_i) = \cup f(A_i)$.

Fuzzy Topological Subalgebra

definition 3.1. The eleventh point is a K-algebra fuzzy topology τ . An indiscrete fuzzy topology is K if and only if it contains two types of fuzzy sets: empty (\emptyset_K) and entire (1_K). If every fuzzy subset of K is contained in K-algebras with a fuzzy topology τ , we say that it is a discrete fuzzy topology.

definition 3.2. the eleventh The K-algebra and a fuzzy set A K containing a membership function Fuzzy subalgebras of K are defined for any x and y in K, $\mu_A(x \cdot y)$ is greater than or equal to the minimum of $\mu_A(x)$ and $\mu_A(y)$.

Example 3.3. [9] Consider the K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3, a^4\}$ is the cyclic group of order 5 and \odot is given by the following Cayley table:

\odot	e	a	a^2	a^3	a^4
e	e	a^4	a^3	a^2	a
a	a	e	a^4	a^3	a^2
a^2	a^2	a	e	a^4	e^3
a^3	a^3	a^2	a	e	e^4
a^4	a^4	a^3	a^2	a	e

(a) If we define a fuzzy set as follows:

$$A = \langle x, \left(\frac{e}{0.5}, \frac{a}{0.4}, \frac{a^2}{0.3}, \frac{a^3}{0.3}, \frac{a^4}{0.2} \right) \rangle$$

$$B = \langle x, \left(\frac{e}{0.6}, \frac{a}{0.4}, \frac{a^2}{0.5}, \frac{a^3}{0.5}, \frac{a^4}{0.3} \right) \rangle$$

Then the family $\{\emptyset_{\mathcal{K}}, 1_{\mathcal{K}}, A, B\}$ of fuzzy sets in \mathcal{K} is a fuzzy topology on \mathcal{K} because the empty fuzzy set $\emptyset_{\mathcal{K}}$ and the whole fuzzy set $1_{\mathcal{K}}$ are in τ , and the intersection of any two members of τ is a member of τ , and arbitrary union of members of τ is a member of τ .

(b) Define a fuzzy set A in \mathcal{K} with membership function μ_A defined by $\mu_A(e) = 0.8$ and $\mu_A(x) = 0.02$ for all $x \neq e$ in \mathcal{K} . It is easy to check that A is a fuzzy subalgebra of \mathcal{K} .

K. H. Dar and M. Akram revealed the following findings in their publication [11].

Definition 3.4. Let (F_1, τ_{F_1}) and $(F_2, \tau_{F_2}^*)$ be fuzzy subspaces of fuzzy topological spaces (\mathcal{K}_1, τ) and (\mathcal{K}_2, τ^*) respectively, and let f be a mapping from (\mathcal{K}_1, τ) to (\mathcal{K}_2, τ^*) . Then f is a mapping of (F_1, τ_{F_1}) into $(F_2, \tau_{F_2}^*)$ if $f(F_1) \subset F_2$. Furthermore, f is *relatively fuzzy continuous* if for each open fuzzy set V_{F_2} in τ_{F_2} , the intersection $f^{-1}(V_{F_2}) \cap F_1$ is in τ_{F_1} .

Moreover, f is *relatively fuzzy open* if for each open fuzzy set U_{F_1} in τ_{F_1} , the image $f(U_{F_1})$ is in τ^* .

Theorem 3.5. *Let \mathcal{K}_1 and \mathcal{K}_2 be K -algebras and let (F, τ_F) and (G, τ_G) be fuzzy subspaces of (\mathcal{K}_1, τ) and (\mathcal{K}_2, τ^*) respectively. Let f be a fuzzy continuous mapping of \mathcal{K}_1 into \mathcal{K}_2 such that $f(F_1) \subset F_2$. Then f is relatively fuzzy continuous mapping of F_1 into F_2 .*

Proof. Let V_{F_2} be a fuzzy set in $\tau_{F_2}^*$, then there exists $V \in \tau^*$ such that $V_{F_2} = V \cap F_2$. Since f is fuzzy continuous, it follows that $f^{-1}(V)$ is a fuzzy set in τ .

Hence $f^{-1}(V_{F_2}) \cap F_1 = f^{-1}(V \cap F_2) \cap F_1 = f^{-1}(V) \cap f^{-1}(F_2) \cap F_1 = f^{-1}(V) \cap F_1$ is a fuzzy set in τ_{F_1} . This completes the proof. \square

Definition 3.6. Let τ_1 and τ_2 be fuzzy topologies on K -algebras \mathcal{K}_1 and \mathcal{K}_2 respectively and A be a fuzzy set with membership function μ_A . A function $f : (\mathcal{K}_1, \tau_1) \rightarrow (\mathcal{K}_2, \tau_2)$ is said to be a *fuzzy continuous map* from (\mathcal{K}_1, τ_1) to (\mathcal{K}_2, τ_2) if it satisfies following conditions:

- (i) For every $A \in \tau_2$, $f^{-1}(A) \in \tau_1$,
- (ii) For every fuzzy subalgebras A (of \mathcal{K}_2) in τ_2 , $f^{-1}(A)$ is a fuzzy subalgebra (of \mathcal{K}_1) in τ_1 .

Theorem 3.7. *If τ_1 is a fuzzy topology on the K -algebra \mathcal{K}_1 and τ_2 is an indiscrete fuzzy topology on the K -algebras \mathcal{K}_2 , then every function $f : (\mathcal{K}_1, \tau_1) \rightarrow (\mathcal{K}_2, \tau_2)$ is a fuzzy continuous map.*

Proof. Since τ_2 is an indiscrete fuzzy topology, therefore, $\tau_2 = \{\emptyset_{\mathcal{K}_2}, 1_{\mathcal{K}_2}\}$.

Let $f : (\mathcal{K}_1, \tau_1) \rightarrow (\mathcal{K}_2, \tau_2)$ be any mapping of K -algebras. We see that every member of τ_2 is a fuzzy subalgebra of K -algebra \mathcal{K}_2 . So it is enough to prove that for every $A \in \tau_2$, $f^{-1}(A) \in \tau_1$.

Let $\emptyset_{\mathcal{K}_2} \in \tau_2$, then for any $x \in \mathcal{K}_1$,

Let $\emptyset_{\mathcal{K}_2} \in \tau_2$, then for any $x \in \mathcal{K}_1$,

$$\begin{aligned} f^{-1}(\emptyset_{\mathcal{K}_2})(x) &= \emptyset_{\mathcal{K}_2}(f(x)) \\ &= 0 \text{ [as } f(x) \in \mathcal{K}_2] \\ &= \emptyset_{\mathcal{K}_1}(x) \text{ [by definition of empty fuzzy set]}. \end{aligned}$$

Thus $(f^{-1}(\emptyset_{\mathcal{K}_2})) = \emptyset_{\mathcal{K}_1} \in \tau_1$.

On the other hand, if $1_{\mathcal{K}_2} \in \tau_2$ and $x \in \mathcal{K}_1$, then we have

$$\begin{aligned} (f^{-1}(1_{\mathcal{K}_2}))(x) &= 1_{\mathcal{K}_2}(f(x)) \\ &= 1 \text{ [as } f(x) \in \mathcal{K}_2] \\ &= 1_{\mathcal{K}_1}(x) \text{ [by definition of whole fuzzy set]}. \end{aligned}$$

Thus $(f^{-1}(1_{\mathcal{K}_2})) = 1_{\mathcal{K}_1} \in \tau_1$. Hence f is a fuzzy continuous map which maps (\mathcal{K}_1, τ_1) into (\mathcal{K}_2, τ_2) . \square

Theorem 3.8. *Let τ_1 and τ_2 be any two discrete fuzzy topologies on the K -algebras \mathcal{K}_1 and \mathcal{K}_2 respectively. Then every homomorphism $f : (\mathcal{K}_1, \tau_1) \rightarrow (\mathcal{K}_2, \tau_2)$ is a fuzzy continuous map.*

Proof. Since τ_1 and τ_2 are discrete fuzzy topologies on K -algebras \mathcal{K}_1 and \mathcal{K}_2 respectively, for every $A \in \tau_2$, $f^{-1}(A) \in \tau_1$ [Note that f is not the usual inverse homomorphism from \mathcal{K}_2 to \mathcal{K}_1].

Let A be a fuzzy subalgebra in τ_2 with membership function μ_A , then for any $x, y \in \mathcal{K}_1$, we have

$$\begin{aligned} (f^{-1}(\mu_A))(x \odot y) &= \mu_A(f(x \odot y)) \\ &= \mu_A(f(x) \odot f(y)) \text{ [since } f \text{ is homomorphism]} \\ &\geq \min\{\mu_A(f(x)), \mu_A(f(y))\} \text{ [for } \mu_A \text{ is a fuzzy subalgebra of } \mathcal{K}_2] \\ &= \min\{(f^{-1}(\mu_A))(x), (f^{-1}(\mu_A))(y)\}. \end{aligned}$$

This shows that $f^{-1}(A)$ is a fuzzy subalgebra (of \mathcal{K}_1) in τ_1 and hence f is a fuzzy continuous map from \mathcal{K}_1 to \mathcal{K}_2 . \square

Theorem 3.9. *Let τ_1 and τ_2 be two fuzzy topologies defined on the K -algebras \mathcal{K}_1 and \mathcal{K}_2 , respectively. Then every homomorphism $f : (\mathcal{K}_1, \tau_1) \rightarrow (\mathcal{K}_2, \tau_2)$ need not in general be a fuzzy continuous map.*

Proof. To prove this theorem it is sufficient if we prove the result to be false for a particular τ_1 and τ_2 defined on any K -algebra \mathcal{K} as in our definition of a fuzzy continuous map we have not assumed \mathcal{K}_1 and \mathcal{K}_2 to be distinct.

Let \mathcal{K} be any K -algebra. Define two fuzzy topologies τ_1 and τ_2 on K -algebra \mathcal{K} as $\tau_1 = \{0_{\mathcal{K}}, 1_{\mathcal{K}}, \mu\}$ and $\tau_2 = \{0_{\mathcal{K}}, 1_{\mathcal{K}}, \lambda\}$, where $\mu, \lambda : \mathcal{K} \rightarrow [0, 1]$ defined as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = e \\ 0 & \text{if } x \neq e \end{cases}$$

and

$$\lambda(x) = \begin{cases} 1 & \text{if } x \neq e \\ 0 & \text{if } x = e \end{cases}$$

where e is identity of a K -algebra. Define $f : (\mathcal{K}, \tau_1) \rightarrow (\mathcal{K}, \tau_2)$ by $f(x) = x$ for all $x \in \mathcal{K}$. Clearly, f is a homomorphism. For $x \in \mathcal{K}$ and $\lambda \in \tau_2$, we have

$$(f^{-1}(\lambda))(x) = \lambda(f(x)) = \lambda(x)$$

This gives $(f^{-1}(\lambda))(x) = \lambda(x)$, for all $x \in \mathcal{K}$. That is $f^{-1}(\lambda) = \lambda$. Thus $f^{-1}(\lambda) \notin \tau_1$, as $\lambda \notin \tau_1$. Hence f is not a fuzzy continuous map on \mathcal{K} . \square

Definition 3.10. Let (\mathcal{K}_1, τ_1) and (\mathcal{K}_2, τ_2) be any two fuzzy topological spaces. A function $f : (\mathcal{K}_1, \tau_1) \rightarrow (\mathcal{K}_2, \tau_2)$ is said to be a *fuzzy homomorphism* if it satisfies the following conditions:

- f is bijective,
- both f and f^{-1} are fuzzy continuous maps.

Definition 3.11. A fuzzy topological space (\mathcal{K}, τ) is said to be a *fuzzy Hausdorff space* if and only if for any two distinct fuzzy points x and y there exist open fuzzy sets F and G such that $F \cap G = \emptyset_{\mathcal{K}}$.

Theorem 3.12. *Let τ_1 and τ_2 be the topologies on K -algebras \mathcal{K}_1 and \mathcal{K}_2 respectively and let $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be a fuzzy homeomorphism. Then \mathcal{K}_1 is a fuzzy Hausdorff space if and only if \mathcal{K}_2 is a fuzzy Hausdorff space.*

Proof. Suppose that \mathcal{K}_1 is a fuzzy Hausdorff space. Let x_t and x_s be the fuzzy points in τ_2 with $x \neq y (x, y \in \mathcal{K}_1)$, then $f^{-1}(x) \neq f^{-1}(y)$, as f is one to one.

For $z \in X$, we consider

$$\begin{aligned}
(f^{-1}(x_1))(z) = x_1(f(z)) &= \begin{cases} t \in (0, 1], & \text{if } f(z)=x; \\ 0, & \text{if } f(z) \neq x. \end{cases} \\
&= \begin{cases} t \in (0, 1], & \text{if } z = f^{-1}(x); \\ 0, & \text{if } z \neq f^{-1}(x). \end{cases} \\
&= (f^{-1}(x))_1(z).
\end{aligned}$$

That is, $(f^{-1}(x_t))(z) = (f^{-1}(x))_t(z)$, for all $z \in \mathcal{K}_1$.

Hence, we have $f^{-1}(x_t) = (f^{-1}(x))_t$.

Similarly we can prove that $f^{-1}(x_s) = (f^{-1}(x))_s$.

By definition of a fuzzy Hausdorff space, there exist open fuzzy sets F_x and G_y of $f^{-1}(x_t)$ and $f^{-1}(x_s)$ respectively such that $F_x \cap G_y = \emptyset_{\mathcal{K}_1}$. Since f is a fuzzy continuous map from \mathcal{K}_1 to \mathcal{K}_2 and f^{-1} is a fuzzy continuous map from \mathcal{K}_2 to \mathcal{K}_1 , there exist open fuzzy sets $f(F_x)$ and $f(G_y)$ of x_t and y_s respectively such that $f(F_x) \cap f(G_y) = f(F_x \cap G_y) = f(\emptyset_{\mathcal{K}_1}) = \emptyset_{\mathcal{K}_2}$. Hence \mathcal{K}_2 is a fuzzy Hausdorff space.

Conversely, let (\mathcal{K}_2, τ_2) be a fuzzy Hausdorff space. By a similar argument and by also using the fact that both f and f^{-1} are fuzzy continuous maps we can prove that (\mathcal{K}_1, τ_1) is a fuzzy Hausdorff space. The proof is now completed. \square

Definition 3.13. Let τ be a fuzzy topology on a K -algebra \mathcal{K} . A FTS (\mathcal{K}, τ) is said to be a *fuzzy C_5 - disconnected* if there exists a fuzzy open and fuzzy closed set F with membership μ_F such that $\mu_F \neq 1_{\mathcal{K}}$ and $\mu_F \neq \emptyset_{\mathcal{K}}$. A FTS (\mathcal{K}, τ) is said to be a *fuzzy C_5 - connected* if it is not a fuzzy C_5 -disconnected.

Theorem 3.14. Let τ_1 and τ_2 be the fuzzy topologies on $K(G)$ -algebras \mathcal{K}_1 and \mathcal{K}_2 respectively. Let $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be a fuzzy continuous surjective mapping. If (\mathcal{K}_1, τ_1) is a fuzzy C_3 -connected, then (\mathcal{K}_2, τ_2) is a fuzzy C_3 -connected.

Proof. Assume that (\mathcal{K}_2, τ_2) is a fuzzy C_3 -disconnected. Then there exist a fuzzy open and closed set F with membership function μ_F such that $\mu_F \neq 1_{\mathcal{K}_2}$ and $\mu_F \neq \emptyset_{\mathcal{K}_2}$. Since f is a fuzzy continuous mapping, $f^{-1}(\mu_F)$ is both OFS and CFS.

Thus $f^{-1}(\mu_F) = 1_{\mathcal{K}_1}$ or $f^{-1}(\mu_F) = \emptyset_{\mathcal{K}_1}$ which is impossible. [since $\mu_F = f(f^{-1}(\mu_F))$, $f(1_{\mathcal{K}_1}) = 1_{\mathcal{K}_2}$, and $\mu_F = f(f^{-1}(\mu_F)) = f(\emptyset_{\mathcal{K}_1}) = \emptyset_{\mathcal{K}_2}$.]

This is contradiction to our assumption. Hence (\mathcal{K}_2, τ_2) is also a fuzzy C_3 -connected. \square

Definition 3.15. Let τ be a fuzzy topology on a K -algebra \mathcal{K} and F be a fuzzy set in \mathcal{K} with membership function μ_F . If a class $\{ \langle x, \mu_{F_i} \rangle : i \in I \}$ of OFS in \mathcal{K} satisfies the condition $F \subseteq \cup \{ \langle x, \mu_{F_i} \rangle : i \in I \}$, then it is called a *fuzzy open cover* of F .

A finite subclass of the fuzzy open cover $\{ \langle x, \mu_{F_i} \rangle : i \in I \}$ of F , which is also a fuzzy open cover of F , is called a *finite subcover* of $\{ \langle x, \mu_{F_i} \rangle : i \in I \}$. A FS $F = \langle x, \mu_F \rangle$ in a FTS (X, τ) is called a *fuzzy compact*, if every fuzzy open cover of F has a finite subcover.

Theorem 3.16. Let τ_1 and τ_2 be the fuzzy topologies on K -algebras \mathcal{K}_1 and \mathcal{K}_2 respectively. Let $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be a fuzzy continuous mapping. If F is a fuzzy compact in (\mathcal{K}_1, τ_1) , then $f(F)$ is a fuzzy compact in (\mathcal{K}_2, τ_2) .

Proof. Let $\mathcal{A} = \{ \mu_{F_i} : i \in I \}$, where $\mu_{F_i} = \langle y, \mu_{F_i} \rangle$ be a fuzzy open cover of $f(F)$. Then $\mathcal{B} = \{ f^{-1}(\mu_{F_i}) : i \in I \}$ is a fuzzy open cover of F . Since F is a fuzzy compact, there exists a finite subcover $\mu_{F_i} (i = 1, 2, \dots, n)$ of F such that $F \subseteq \cup_{i=1}^n f^{-1}(\mu_{F_i})$. Thus

$$f(F) \subseteq f(\cup_{i=1}^n f^{-1}(\mu_{F_i}))$$

$$f(F) \subseteq \cup_{i=1}^n f(f^{-1}(\mu_{F_i}))$$

$$f(F) \subseteq \cup_{i=1}^n \mu_{F_i}$$

follows. Hence $f(F)$ is a fuzzy compact in (\mathcal{K}_2, τ_2) . \square

Theorem 3.17. Let \mathcal{K} be a K -algebra and let τ be a fuzzy topology on K -algebra \mathcal{K} . Let A be a fuzzy topological algebra in \mathcal{K} . Then the inversion map $f : A \rightarrow A$ defined by $f(x) = x^{-1}$ and the inner automorphism $h : A \rightarrow A$ defined by $h(g) = aga^{-1}$ are all relative fuzzy homomorphisms, where $a \in \{ x : A(x) = A(e) \}$.

Proof. Clearly f is one-to-one. Since

$$f(A)(y) = \sup_{z \in f^{-1}(y)} A(z) = A(y^{-1}) = A(y) \quad \forall y \in A.$$

That is, $f(A) = A$. Since $f^{-1}(x) = x^{-1}$ is relatively fuzzy continuous, f is relatively fuzzy open. Thus f is a relative fuzzy homomorphism. Let $r_a : A \rightarrow A$ be a right translation defined by $r_a(x) = xa$ and $l_a : A \rightarrow A$ be left translation defined by $l_a(x) = ax$. Then

$$\begin{aligned} (r_a(A))(x) &= \sup_{z \in r_a^{-1}(x)} A(z) = A(xa^{-1}) \\ &\geq \min(A(x), A(a^{-1})) = \min(A(x), A(e)) \\ &= A(x) = A(xa^{-1}a) \\ &\geq \min(A(xa^{-1}), A(a)) \\ &= A(xa^{-1}) = (r_a(A))(x). \end{aligned}$$

Thus $r_a(A) = A$. Let $\phi : A \rightarrow A \times A$ be a map defined by $\phi(x) = (x, a)$ and $\psi : A \times A \rightarrow A$ be a map defined by $\psi(x, y) = xy$. Then $r_a = \psi \circ \phi$. Since ϕ and ψ are fuzzy continuous, r_a is fuzzy continuous. Since $r_a^{-1} = r_{a^{-1}}$, r_a is a fuzzy homomorphism. Similarly, l_a is a fuzzy homomorphism. Since h is a composition of $r_{a^{-1}}$ and l_a , h is relatively fuzzy homomorphism. This completes the proof. □

4 Fuzzy topological ideals

Definition 4.1. A fuzzy set A in a K -algebra \mathcal{K} with membership function μ_A is called a *fuzzy ideals* of \mathcal{K} if it satisfies:

- (i) $(\forall x \in G) (\mu_A(e) \geq \mu_A(x))$,
- (ii) $(\forall x, y \in G) (\mu_A(x) \geq \min\{\mu_A(x \odot y), \mu_A(y \odot (y \odot x))\})$.

Example 4.2. [9] Consider the $K(S_3)$ -algebra $\mathcal{K} = (S_3, \cdot, \odot, e)$ on the symmetric group $S_3 = \{e, a, b, x, y, z\}$ where $e = (1)$, $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, and \odot is given by the following Cayley table:

\odot	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

Define a fuzzy set A in \mathcal{K} with membership function μ_A by $\mu_A(e) = t_1, \mu_A(p) = t_2$, for all $p \neq e$, where $t_1 > t_2$ in $[0, 1]$. It is easy to check that A is a fuzzy ideal of \mathcal{K} .

Definition 4.3. Let \mathcal{K} be a K -algebra and τ a fuzzy topology on \mathcal{K} . Let A be a fuzzy K -algebra with induced fuzzy topology τ_A . Then A is called a fuzzy topological K -algebra if for each $a \in \mathcal{K}$ the self mapping $r_a : (A, \tau_A) \rightarrow (A, \tau_A)$ defined by

$$r_a(g) = g \odot a \quad \forall g \in \mathcal{K}, \quad (1)$$

is relatively fuzzy continuous.

Theorem 4.4. Let $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be a homomorphism of K -algebras. Let τ and τ^* be the fuzzy topologies on \mathcal{K}_1 and \mathcal{K}_2 respectively such that $\tau = f^{-1}(\tau^*)$. If B is a fuzzy topological ideal of \mathcal{K}_2 with membership function μ_B . Then $f^{-1}(B)$ is a fuzzy topological ideal of \mathcal{K}_1 with membership function $\mu_{f^{-1}(B)}$.

Proof. It is easy to show that $\mu_{f^{-1}(B)}(e) \geq \mu_{f^{-1}(B)}(x)$, for all $x \in \mathcal{K}_1$. For any $x, y \in \mathcal{K}_1$, we have

$$\begin{aligned} \mu_{f^{-1}(B)}(x) &= \mu_B(f(x)) \\ &\geq \min\{\mu_B(f(x \odot y)), \mu_B(f(y \odot (y \odot x)))\} \\ &\geq \min\{\mu_B(f(x) \odot f(y)), \mu_B(f(y) \odot f(y \odot x))\} \\ &= \min\{\mu_{f^{-1}(B)}(x \odot y), \mu_{f^{-1}(B)}(y \odot (y \odot x))\}. \end{aligned}$$

Hence $f^{-1}(B)$ is a fuzzy ideal of \mathcal{K}_1 .

Let F_1 be an open fuzzy set in $\tau_{f^{-1}(B)}$ on $f^{-1}(B)$. Since f is a fuzzy continuous mapping of (\mathcal{K}_1, τ) into (\mathcal{K}_2, τ^*) , f is a relatively fuzzy continuous mapping of $(f^{-1}(B), \tau_{f^{-1}(B)})$ into (B, τ_B^*) . If there exists open fuzzy set $F_2 \in \tau_B^*$ such that

$$f^{-1}(F_2) = F_1 \quad (2)$$

Then

$$\begin{aligned} \mu_{r_a^{-1}(F_1)}(g) &= \mu_{F_1}(r_a(g)) = \mu_{F_1}(g \odot a) \\ &= \mu_{f^{-1}(F_2)}(g \odot a) = \mu_{F_2}(f(g \odot a)) \\ &= \mu_{F_2}(f(g) \odot f(a)) \end{aligned}$$

$$\mu_{r_a^{-1}(F_1)}(g) = \mu_{F_2}(f(g) \odot f(a)) \quad (3)$$

Since B is a fuzzy topological in \mathcal{K}_2 , the mapping $r_b : (B, \tau_B) \rightarrow (B, \tau_B)$ defined by

$$r_b(y) = y \odot b \quad \forall b \in \mathcal{K}_2 \quad (4)$$

is relatively fuzzy continues.

$$\begin{aligned}\mu_{r_a^{-1}(F_1)}(g) &= \mu_{F_2}(f(g) \odot f(a)) = \mu_{F_2}(r_{f(a)}(f(g))) \\ &= \mu_{r^{-1}f(a)(F_2)}(f(g)) = \mu_f^{-1}(r_{f(a)}^{-1}(F_2))(g)\end{aligned}$$

This implies that $r_a^{-1}(F_1) = f^{-1}(r_{f(a)}^{-1}(F_2))$ and

so $r_a^{-1}(F_1) \cap f^{-1}(B) = f^{-1}(r_{f(a)}^{-1}(F_2)) \cap f^{-1}(B)$ is open in the induced fuzzy topology on $f^{-1}(B)$. □

Theorem 4.5. *Let $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be an isomorphism of K -algebras. Let τ and τ^* be the fuzzy topologies on \mathcal{K}_1 and \mathcal{K}_2 respectively such that $f(\tau) = \tau^*$. If A is a fuzzy topological ideal of \mathcal{K}_1 with membership function μ_A . Then $f(A)$ is a fuzzy topological ideal of \mathcal{K}_2 with membership function $\mu_{f(A)}$.*

Proof. It is easy to show that $\mu_{f(A)}(e) \geq \mu_{f(A)}(x)$, for all $x \in \mathcal{K}_2$.

Given $x, y \in \mathcal{K}_2$, let $x_0 \in f^{-1}(x)$, $y_0 \in f^{-1}(y)$ such that $\mu_A(x_0) = \sup_{t \in f^{-1}(x)} \mu_A(t)$, $\mu_A(y_0) = \sup_{t \in f^{-1}(y)} \mu_A(t)$. Then, we have

$$\begin{aligned}\mu_{f(A)}(x) &= \sup_{t \in f^{-1}(x)} \mu_A(t) \\ &\geq \mu_A(x_0) \\ &\geq \mu_A(x_0) \\ &\geq \min\{\mu_A(x_0 \odot y_0), \mu_A(y_0 \odot (y_0 \odot x_0))\} \\ &= \min\left\{\sup_{t \in f^{-1}(x \odot y)} \mu_A(t), \sup_{t \in f^{-1}(y \odot (y \odot x))} \mu_A(t)\right\} \\ &= \min\{\mu_{f(A)}(x \odot y), \mu_{f(A)}(y \odot (y \odot x))\}.\end{aligned}$$

Hence $f(A)$ is a fuzzy ideal of \mathcal{K}_2 .

Now we show that the mapping $r_b : (f(A), \tau_{f(A)}^*) \rightarrow (f(A), \tau_{f(A)}^*)$ defined by

$$r_b(y) = y \odot b \tag{5}$$

is relatively fuzzy continuous for each $b \in \mathcal{K}_2$. Let U_A be a fuzzy set in τ_A . Then there exists a fuzzy set U in τ such that $U_A = U \cap A$. Since f is one-one, it follows that $f(U_A) = f(U \cap A) = f(U) \cap f(A)$ which is a fuzzy set in $\tau_{f(A)}^*$. This shows that f is relatively fuzzy open.

Let $V_{f(A)}$ be a fuzzy set in $\tau_{f(A)}^*$. The onto mapping of f implies that for each $b \in \mathcal{K}_2$, there exists $a \in \mathcal{K}_1$ such that

$$b = f(a) \tag{6}$$

Thus

$$\mu_{f^{-1}(r_b^{-1}(V_{f(A)}))}(x) = \mu_{f^{-1}(r_{f(a)}^{-1}(V_{f(A)}))}(x) = \mu_{r_{f(a)}^{-1}(V_{f(A)})}(f(x))$$

$$\begin{aligned}
\mu_{f^{-1}(r_b^{-1}(V_{f(A)}))}(x) &= \mu_{f^{-1}(r_{f(a)}^{-1}(V_{f(A)}))}(x) = \mu_{r_{f(a)}^{-1}(V_{f(A)})}(f(x)) \\
&= \mu_{V_{f(A)}}(r_{f(a)}(f(x))) = \mu_{V_{f(A)}}(f(x) \odot f(a)) \\
&= \mu_{f^{-1}(V_{f(A)})}(x \odot a) = \mu_{f^{-1}(V_{f(A)})}(r_a(x)) \\
&= \mu_{r_a^{-1}(f^{-1}(V_{f(A)}))}(x).
\end{aligned}$$

This implies that $f^{-1}(r_a^{-1}(V_{f(A)})) = r^{-1}(f^{-1}(V_{f(A)}))$. The mapping $r_a : (A, \tau_A) \rightarrow (A, \tau_A)$ defined by $r_a(x) = x \odot a$ is relatively fuzzy continuous and f is relatively fuzzy continuous map $(A, \tau_A) \rightarrow (f(A), \tau_{f(A)}^*)$.

Hence $f^{-1}(r_b^{-1}(V_{f(A)})) \cap A = r_a^{-1}(f^{-1}(V_{f(A)})) \cap A$ is a fuzzy set in τ_A . \square

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