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Article

Analytical Methods for Solving Mathematical Systems of Ordinary Differential Equations

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Abstract: This study addresses the analytical resolution of systems of ordinary differential equations (ODEs), which are foundational in modeling various dynamic processes across scientific fields. While numerous methods exist, a clear comparative framework for solving linear systems remains underexplored. This paper fills that gap by employing and contrasting three core techniques: the D-operator method, eigenvalue analysis, and integral transforms (especially Laplace). Each method is applied to illustrative examples, demonstrating their efficiency, limitations, and the conditions under which they yield general solutions. The results reveal that integral transforms, particularly Laplace, offer more streamlined solutions for linear systems with initial conditions, while eigenvalue methods excel in homogeneous cases. These findings provide valuable insights for selecting appropriate analytical tools in mathematical modeling and engineering applications.

Keywords: ordinary differential equations, d-operator method, eigenvalue method, laplace transform, analytical solution, linear systems

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1. Introduction

A differential equation is defined as an equation that includes derivatives of unknown functions concerning one or more variables. The primary purpose of a differential equation is to articulate the relationship between a function and its derivatives, and it finds extensive application across various scientific disciplines including physics, engineering, economics, and more.[1]

The mathematical framework of differential equations consists of a collection of equations that delineate the relationships between mathematical functions and their derivatives in relation to independent variables. These equations are employed to model natural phenomena or processes that undergo changes over time or in response to other variables.[2]

Accurate solutions to differential equations offer a means to ensure that mathematical models authentically represent reality, thereby facilitating more effective and precise applications across numerous fields.[3]

2. Materials and Methods

The method of such study is based on analytical mathematical means to solve systems of ordinary differential equations (ODEs). The research started with the general linear systems of first and higher order ODEs of both standard as well as matrix form. These formulations were then used as a basis for applying in turn, the D-operator method, the eigenvalue method and a set of integral transforms. In the D-operator approach, differential equations were operated on in terms of operator notation, so as to have a systematic elimination of variables and to find characteristic equations involving arbitrary constants producing general solutions. In this method, the system was first converted to matrix form and then eigenvalues and associated eigenvectors were obtained in order to express general solution as a linear combination of fundamental solutions. Specifically, this method performed particularly well in treating homogeneous linear systems. The third technique involved the use of integral transforms in particular, the Laplace transform. This method transformed the differential system to algebraic system in the domain of Laplace, and solved it with algebraic method using partial fraction decomposition. In that case, the time domain solution was then obtained by applying the inverse Laplace transform. Carefully built examples were used to demonstrate each method and point out its procedural steps, when applicable, and its effectiveness. Their efficiency, solution uniqueness and practical relevance were compared across methods. Finally, it shows that the classical tools are still valid and offers a systematic methodology to determine what techniques will be best suited to study differential systems according to their nature and complexity.

3. Results and Discussion

The Formula of General System of Ordinary Differential Equation(SODE) First Order [4].

Linear non-homogenous (SODE)of first-order has the form:

$$Y_1' = \mathcal{K}_{11} Y_1(t) + \mathcal{K}_{12} Y_2(t) + \cdots + \mathcal{K}_{1n} Y_n(t) + a_1(t)$$
$$Y_2' = \mathcal{K}_{21} Y_1(t) + \mathcal{K}_{22} Y_2(t) + \cdots + \mathcal{K}_{2n} Y_n(t) + a_2(t)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$Y_n' = \mathcal{K}_{n1} Y_1(t) + \mathcal{K}_{n2} Y_2(t) + \cdots + \mathcal{K}_{nn} Y_n(t) + a_n(t),$$

or in a matrix form

$$Y'(t) = \mathcal{K} Y(t) + a(t). \qquad (1.1)$$

Where

$$Y'(t) = \begin{pmatrix} \frac{dY_1(t)}{dt} \\ \frac{dY_2(t)}{dt} \\ \vdots \\ \frac{dY_n(t)}{dt} \end{pmatrix}, \ \mathcal{K} = (\mathcal{K}ij) = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \dots & \mathcal{K}_{1n} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \dots & \mathcal{K}_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \mathcal{K}_{n1} & \mathcal{K}_{n2} & \dots & \mathcal{K}_{nn} \end{pmatrix},$$

$$Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_n(t) \end{pmatrix}, \qquad a(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{pmatrix}$$

If $a(t) = 0$, then the system (1.1) is called ahomogenous system.

3. The General System of r-Order Formula was extended in n dimensions [5].

A linear system of r-order in n-dimensional space is expressed as follows:

$$Y_1^{(r)}(t) = \mathcal{K}_{11} Y_1(t) + \mathcal{K}_{12} Y(t) + \cdots + \mathcal{K} Y_n(t) + a(t),$$
$$Y_2^{(r)}(t) = \mathcal{K}_{21} Y_1(t) + k_{22} Y_2(t) + \cdots + \mathcal{K}_{2j} Y_n(t) + a(t),$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$Y_n^{(r)}(t) = \mathcal{K}_{i1} Y_1(t) + \mathcal{K}_{i2} Y_2(t) + \cdots + \mathcal{K}_{ij} Y_n(t) + a_n(t),$$

Where

$$Y^{(r)}(t) = \begin{pmatrix} \frac{d^r Y_1(t)}{dt^r} \\ \frac{d^r Y_2(t)}{dt^r} \\ \vdots \\ \frac{d^r Y_n(t)}{dt^r} \end{pmatrix}, \ \mathcal{K} = (\mathcal{K}ij) = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \dots & \mathcal{K}_{1n} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \dots & \mathcal{K}_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \mathcal{K}_{i1} & \mathcal{K}_{i2} & \dots & \mathcal{K}_{ij} \end{pmatrix},$$

$$Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_n(t) \end{pmatrix}, \qquad a\,(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{pmatrix} \quad ,$$

which can be stated as the following formula:

$$\begin{pmatrix} \frac{d^r Y_1(t)}{dt^r} \\ \frac{d^r Y_2(t)}{dt^r} \\ \vdots \\ \frac{d^r Y_n(t)}{dt^r} \end{pmatrix} = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \dots & \mathcal{K}_{1j} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \dots & \mathcal{K}_{2j} \\ \vdots & \vdots & \dots & \vdots \\ \mathcal{K} & \mathcal{K}_{i2} & \dots & \mathcal{K}_{ij} \end{pmatrix} \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_n(t) \end{pmatrix} + \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a(t) \end{pmatrix},$$

$$Y^r(t) = \mathcal{K}Y(t) + a(t). \tag{1.2}$$

If $\overrightarrow{a(t)} = 0$, then the system(1. 2)called homogenous system.

4. The, Techniques of Linear System Solving [6]

The linear system can be resolved by numerous approaches as:

4.1 The D, Method Operator

D(denotes the derivation, of a specific funct,ion with respect to the independent variable t).

where:$D = \frac{d}{dt}$, $D^2 = \frac{d^2}{dt^2}$, ⋯ , $D^n = \frac{d^n}{dt^n}$, with using elimination variable and simple

The computation might provide the other dependent variables.

Rem,ark 1 [8]: The quantity of arbitrary consta nts that may be present in the general solution of a linear system, such as:

$$\mathcal{K}_{11}(D)Y_{11} + \mathcal{K}_{12}(D)Y_{12} + \cdots + \mathcal{K}_{1n}(D)Y_{1n} = a_1(t)\,,$$
$$\mathcal{K}_{21}(D)Y_{21} + \mathcal{K}_{22}(D)Y_{22} + \cdots + \mathcal{K}_{2n}(D)Y_{2n} = a_2(t),$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$\mathcal{K}_{m1}(D)Y_{m1} + \mathcal{K}_{m2}(D)Y_{m2} + \cdots, \mathcal{K}_{mn}(D)Y_{mn} = a_n(t),$$

where D is an operator which represents $D = \frac{d}{dt}$, $\mathcal{K}_{11}(D) \dots \mathcal{K}_{mn}(D)$ that are fu nctions of D, t andc_j are functions of t , $i = 1,2, \dots, m$ and $j = 1,2, \dots n$ depict dependent an d indep endent var iable in the sy stem, respectively.

They c an be equival ent to the degree of D i n the determinant.

$$\begin{vmatrix} \mathcal{K}_{11}(D) & \mathcal{K}_{12}(D) & \dots & \mathcal{K}_{1n}(D) \\ \mathcal{K}_{21}(D) & \mathcal{K}_{22}(D) & \dots & \mathcal{K}_{2n}(D) \\ \vdots & & & \vdots \end{vmatrix} \equiv \Delta$$
$$\vdots$$

$\mathcal{K}_{m1}(D) \quad \mathcal{K}_{m2}(D) \quad ... \quad \mathcal{K}_{mn}(D)$

If $\Delta \equiv 0$, then the solution set is dependent and outside the scope of o ur analysis; hence, we ass ume Δ is non-ze ro.

Example : the system, resolving.

$$Y_1'(t) - 2Y_1(t) + 2Y_2'(t) = 2 - 4e^{2t} \qquad (1.3)$$

$$2Y_1'(t) - 3Y_1(t) + 3Y_2'(t) - Y_2(t) = 0$$

By the operator D method,

re -write this system in the following form:

$$[(D-2)Y_1(t) + 2DY_2(t) = 2 - 4e^{2t}](3D-1)$$

$$[(2D-3)Y_1(t) + (3D-1)Y_2(t)=0](-2D)$$

We delete (t) get to,

$$(D^2 + D - 2)\, Y_1(t) = -2 - 20e^{2t}$$

(m-1)(m+2)=0 →m=1,m=-2

$$Y_{1c}(t) = \mathcal{K}_1 e^t + \mathcal{K}_2 e^{-2t}$$

$$Y_{1p}(t) = -1 + 5e^{2t}$$

$$Y_1(t) = \mathcal{K}_1 e^t + \mathcal{K}_2 e^{-2t} + 5e^{2t} - 1$$

$$[(D-2)Y_1(t) + 2DY_2(t) = 2 - 4e^{2t}](2D-3)$$

$$[(2D-3)Y_1(t) + (3D-1)Y_2(t)=0](D-2)$$

We delete (t) get to,

$$(D^2 + D - 2)\, Y_2(t) = -6 - 4e^{2t}$$

In the same way above we find.

$$Y_2(t) = \mathcal{K}_3 e^t - \mathcal{K}_4 e^{-2t} - e^{2t} + 3$$

From note (1), the number of consta nts in the general solution exceed the degree of D; so, we insert into (1.3) to obtain

$$\mathcal{K}_3 = \frac{1}{2}\mathcal{K}_1 \quad , \mathcal{K}_4 = -\mathcal{K}_2$$

$$Y_1(t) = Y_1 e^t + Y_2 e^{-2t} + 5e^{2t} - 1,$$

$$Y_2(t) = \frac{1}{2}\mathcal{K}_1 e^t - \mathcal{K}_2 e^{-2t} - e^{2t} + 3\ .$$

4.2 The Eigenvalue Approach

Indices $\lambda_1, \lambda_2, \dots, \lambda_n$ eig en val ues that are uniq ue to the matr i x Y(t) where these values sa tisfy the equation.

$$|Y - \lambda I| = 0. \qquad (1.4)$$

Let $\overrightarrow{b_1}, \overrightarrow{b_2}, \cdots, \overrightarrow{b_n}$ are the eigen vectors achieved by th ese values then the general solution of the system.

$$Y_1(t) = e^{\lambda_1} b_{1,}$$

$$Y_2(t) = e^{\lambda_2} b_2,$$

$$\vdots \qquad \vdots$$

$$Y_n(t) = e^{\lambda_n} b_n.$$

the linear combination of these solution can be expressed as in the following:

$$\mathcal{K}(t) = \mathcal{K}_1 e^{\lambda_1} b_1 + \mathcal{K}_2 e^{\lambda_2} b_2 + \cdots + \mathcal{K}_n e^{\lambda_n} b_n.$$

$\mathcal{K}_1, \mathcal{K}, \cdots, \mathcal{K}_n$ are arbi trary consta nts.

Example : To solve the system.

$$Y_1'(t) = Y_1(t) - Y_2(t) - Y_3(t) \tag{1.5}$$
$$Y_2'(t) = Y_2(t) + 3Y_3(t)$$
$$Y_3'(t) = 3Y_2(t) + Y_3(t)$$

By the eigen value method :

we write this system in the form:

$$\begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{pmatrix}' = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{pmatrix},$$

By formula (1.4) eigenvalues are the roots of the equation

$$\begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)[(\lambda-4)(\lambda+2)] = 0,$$

therefore, one solution is given by the eigenvalue is $\lambda_1 = 1$

$$\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} = 0,$$

$$-\mathcal{K}_1 - \mathcal{K}_3 = 0\,,\ 3\mathcal{K}_3 = 0, 3\mathcal{K}_2 = 0$$

$$Y_1(t) = Y_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t,$$

also, the second eigenval ue $\lambda_2 = 4$, gi ves

$$\begin{pmatrix} -3 & -1 & -1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} = 0$$

$$3\mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 = 0$$

$$-\mathcal{K}_2 + \mathcal{K}_3 = 0 \rightarrow \quad \mathcal{K}_2 = \mathcal{K}_3$$

$$Y_2(t) = \mathcal{K}_2 \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} e^{4t},$$

finally, the last eigenvalue $\lambda_3 = -2$, g ets

$$\begin{pmatrix} 3 & -1 & -1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} = 0$$

$$Y_3(t) = \mathcal{K}_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t},$$

thus ,the general solution of system is :

$$Y(t) = \mathcal{K}_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + \mathcal{K}_2 \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} e^{4t} + \mathcal{K}_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t},$$

where $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ are arbitrary constants.

4.3 Integral Transforms Method

The primary objective of the conversion process is to change the system's shape from differential to algebraic in order to streamline the solution method. While numerous transformations provide the same function, the Laplace transform is particularly useful and efficient when working with linear systems [7]. Accordingly, we derived the mathematical system from it. Systems of ordinary differential equations are amenable to the well-known, precise, and efficient method of integral transformations. There are several integral transformations, including Laplace, Elzaki, Novel, and SEE transformations, among many more. [8]

Definition [9]
The Lapl ace-Carson Transfo rm of the real function $Y(t), t > 0$ is d efined by:

$$L_C(Y(t)) = P\int_0^{\infty} e^{-Pt} Y(t)\ dt, \qquad P > 0$$

L_C is the operator of $L_C T$.

Example : To solve the system.

$$Y_1'(t) - Y_1(t) + Y_2(t) + Y_3(t) = 0 \qquad (1.6)$$

$$Y_2'(t) - Y_2(t) - 3\,Y_3(t) = 0$$

$$Y_3'(t) - 3Y_2(t) - Y_3(t) = 0$$

$$Y_1(0) = Y_2(0) = Y_3(0) = 1$$

Solution:

Us ing form ula (1.2) we have.

$$\Delta = \begin{vmatrix} (P-1) & 1 & 1 \\ 0 & (P-1) & -3 \\ 0 & -3 & (P-1) \end{vmatrix} = (P-1)[(P-1)^2 - 9],$$

$$= (P-1)(P-4)(P+2),$$

$$L(Y_1(t)) = \frac{1}{(P-1)(P-4)(P+2)} \begin{vmatrix} 1 & 1 & 1 \\ 1 & (P-1) & -3 \\ 1 & -3 & (P-1) \end{vmatrix},$$

$$L(Y_1(t)) = \frac{1}{(P-1)(P+2)(P-4)} \left((P-1)^2 - 9) - \left(\frac{(P-1)}{P} + \frac{3}{P}\right) + \left(\frac{-3}{P} - \frac{(P-1)}{P}\right)\right],$$

$$= \frac{1}{(P-1)(P+2)(P-4)} [(P-4)(P+2) - (P+2) - (P+2)],$$

$$= \frac{\mathcal{P}-6}{(P-1)(P-4)},$$

using partition fractions

$$L(Y_1(t)) = \frac{\frac{5}{3}}{(P-1)} - \frac{\frac{2}{3}}{(P-4)},$$

taking the inverse of Laplace transform for both sides of the above equation, to obtain:

$$Y_1(t) = \frac{5}{3}e^{t} - \frac{2}{3}e^{4t}$$

In similar way, $L(Y(t))$ can be obtained by:

$$L(Y_2(t)) = \frac{1}{(P-1)(P-4)(P+2)} \begin{vmatrix} (P-1) & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & (P-1) \end{vmatrix},$$

$$= \frac{1}{(P-1)(P-4)(P+2)} (P-1)(P+2),$$

$$= \frac{P+2}{(P-4)(P+2)} = \frac{1}{(P-4)},$$

also, by the inverse of Laplace transform for the above equation,

$$Y_2(t) = e^{4t}$$

Also,$L(Y_3(t))$ can be obtained by

$$L(Y_3(t)) = \frac{1}{(P-1)(P-4)(P+2)} \begin{vmatrix} (P-1) & 1 & 1 \\ 0 & (P-1) & 1 \\ 0 & -3 & 1 \end{vmatrix},$$

$$L(Y_3(t)) = \frac{1}{(P-4)},$$

the inverse of Laplace transformation for the above equation ,yields:

$$Y_3(t) = e^{4t}.$$

$Y_1(t), Y_2(t)$ and $Y_3(t)$ represent the set solution of the system (1.6).

Because the systems are defined from the outset, the answer that emerges from integral transformations is unique.

4. Conclusion

Consequently, this study systematically investigated how to solve systems of ordinary differential equations using analytical methods, including the D-operator, and eigenvalue, and integral transform approaches and. Each technique is found to be virtually practical for variable elimination in the first order systems, for having structured solutions for linear homogeneous systems, and a powerful tool to convert a differential equation into an algebraic form via integral transform, especially the Laplace transform. The comparison uncovers the fact that all three methods provide mathematically sound solution to the system even with the initial conditions, however, integral transforms are particularly advantageous for systems with initial conditions since they provide unique and exact solution. A number of insights from these examples feed into optimization of mathematical modeling across the scientific and engineering disciplines. The hybridization of these methods or application to non-linear and higher dimensional systems may further be pursued with the aim of increasing computational efficiency and expanding of practical applicability.

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